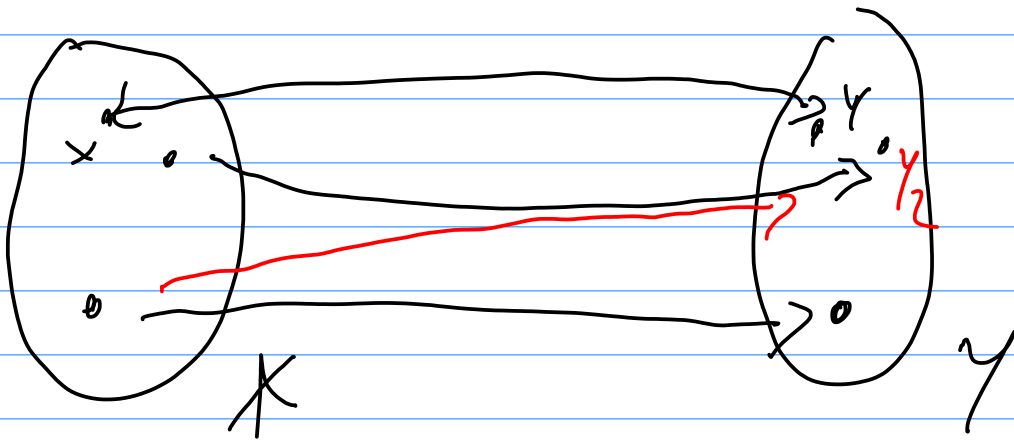
Bijective functions $f: X \to Y$



Surjectivity: A function $f: X \to Y$ is surjective if $\forall y \in Y : \exists x \in X : x \in f^{-1}[\{y\}]$

A bijective function $f: X \to Y$

$\forall y \in Y \ \exists! x \in X : f(x) = y$

$\Rightarrow$ we can define the inverse function

$$f^{-1}: Y \to X$$
$$y \mapsto f^{-1}(y) \in X : f\left(f^{-1}(y)\right) = y$$

Some examples $f: \mathbb{R}^+ \to \mathbb{R}^+$

$x \mapsto x^2$

$f^{-1}: y \mapsto \sqrt{y}$

$f: \mathbb{R} \to \mathbb{R}$

$x \mapsto x + a$

$f^{-1} \; y \mapsto y - a$

We cannot define a inverse for

$f: \mathbb{R} \to \mathbb{R}^+$

$x \mapsto x^2$

$1 \in \mathbb{R}^+ : \quad f(-1) = 1 = f(1)$

$\Rightarrow f$ is not injective,

thus not invertible.

Composition: $h = g \circ f$

$h: X \to Z$

$x \mapsto f(g(x))$

Example $f: X \to Y$ $\quad h: x \mapsto (x+a)^2$

$x = x + a$

$g \; Y \to Z$

$y \to y^2$

Derivatives:

You should know this for univariate real-valued functions!

Some Rules

sum rule $(f(x)+g(x))' = f'(x) + g'(x)$

Product ~~rule~~ $(g(x)\cdot f(x))' = f'(x)g(x) + f(x)\cdot g'(x)$

Quotient rule $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$

Chain rule : $(g(f(x)))' = f'(x) \cdot g'(f(x))$

Limits and Continuity of uni-dimensional Objects

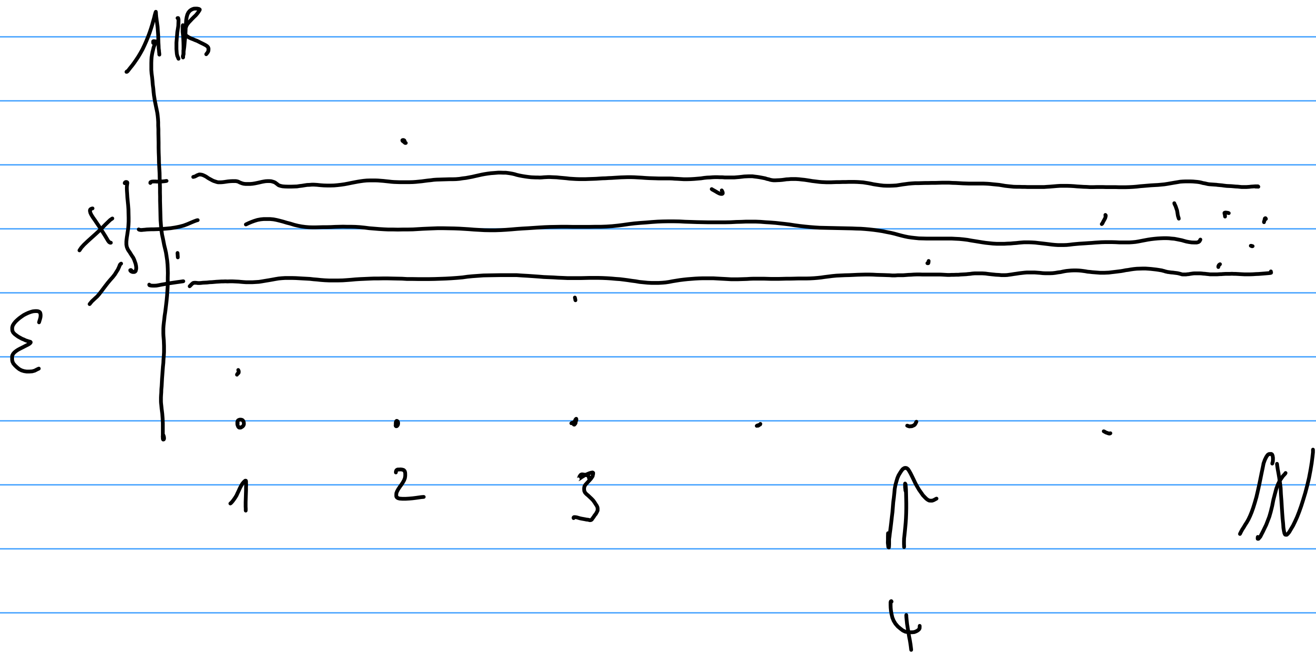
$$\lim_{n\to\infty} \underbrace{\frac{1}{n}}_{\text{sequence}} = 0$$

For a general sequence $\{x_n\}_{n\in\mathbb{N}}$

$x_n \in \mathbb{R} \quad \forall n \in \mathbb{N}$

$x \in \mathbb{R}$ is the limit of $\{x_n\}_{n\in\mathbb{N}}$ if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : \left(\forall n \in \mathbb{N} \ n \geq N \Rightarrow |x_n - x| < \varepsilon \right)$$



We write $\lim\limits_{n\to\infty} x_n = \infty$ if

$$\forall x \in \mathbb{R} \ \exists N \in \mathbb{N} : \forall n \geq N \ \ x_n > x$$

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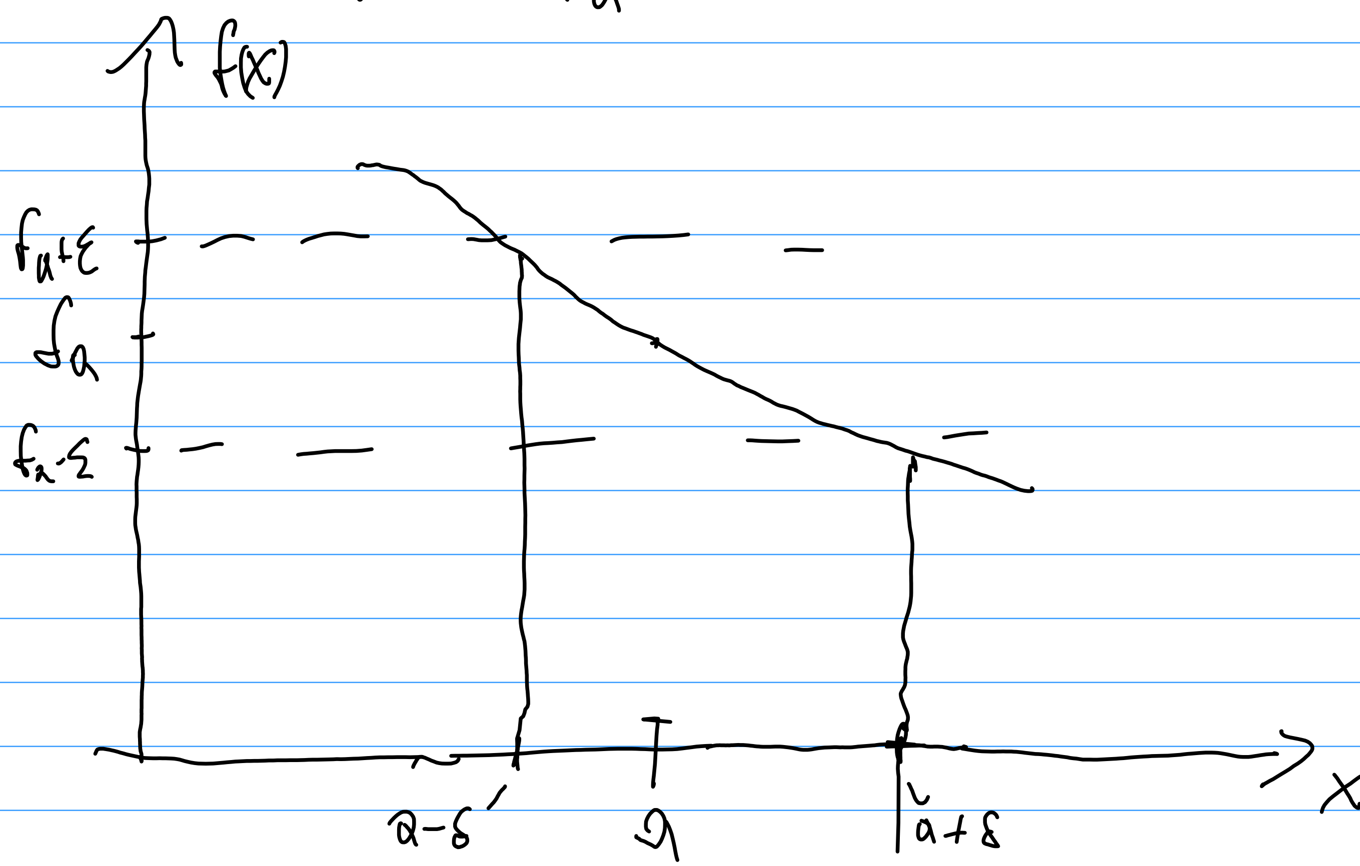
Limit of a function $f: X \to Y$

limit at $a \in \mathbb{R}$

$f_a \in \mathbb{R}$ is called the limit of f at a if

$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in X : (|x-a| \in (0,\delta)$
$\Rightarrow |f(x) - f_a| < \varepsilon)$



$\Rightarrow$ we see: this is related to the continuity of functions

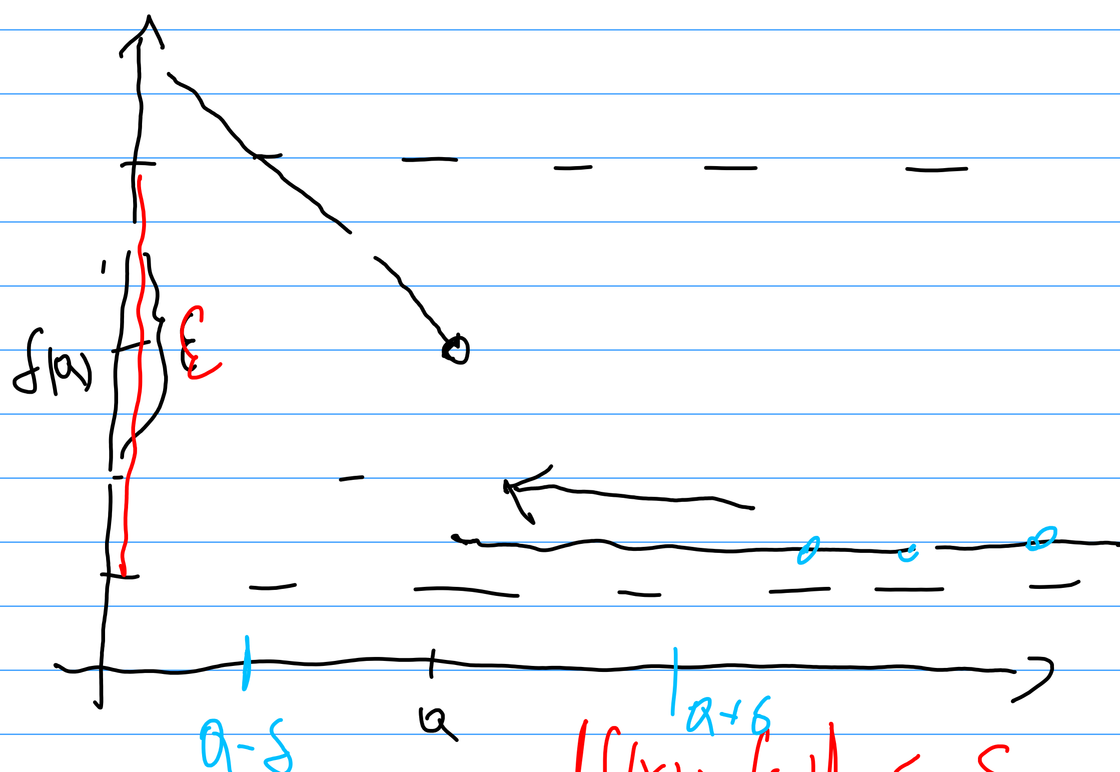
Def: Continuity

$f: X \to Y$ $X, Y \subseteq \mathbb{R}$, f is continuous at $a \in X$ if $\lim_{x \to a} f(x) = f(a)$

This is strong:

$$\lim_{x \to a} f(x) = f\left(\lim_{x \to a} x\right) \text{ iff } f \text{ is continuous}$$

At the discontinuities



$|f(x) - f(a)| < \varepsilon$

The left and the right-limit

$\lim_{x \to a^+} f(x) = f_a^+$ is the right-limit

if

$\forall \varepsilon > 0 \ \exists \delta^+ > 0 : \forall x \in X \quad x - a \in (0, \delta)$

$\Rightarrow f(x) - f(a) < \varepsilon$

Rules : if f is continuous

$$\lim_{x \to a} f(x) = f\left(\lim_{x \to a} x\right)$$

- L'Hospital f, g real-valued functions, differentiable on an open interval I, $a \in I$ or $a \in \{\pm \infty\}$

if $g'(x) \neq 0 \quad \forall x \in I$ and $x \neq a$

Suppose $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

or $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm \infty$

$\Rightarrow$ if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists

it holds that $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Sandwhich - Thm

$\{x_n\}_{n \in \mathbb{N}}$, $\{y_n\}_{n \in \mathbb{N}}$, $\{z_n\}_{n \in \mathbb{N}}$ sequences

$\{y_n\}_{n \in \mathbb{N}}$, $\{z_n\}_{n \in \mathbb{N}}$ convergent

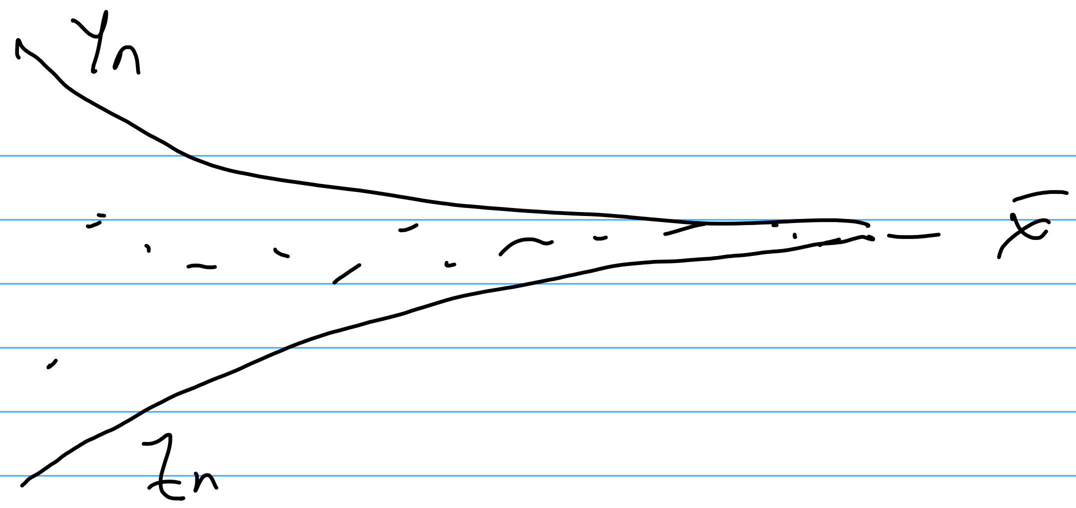
and $\lim_{n \to \infty} y_n = \lim_{n \to \infty} z_n = \bar{x} \in \mathbb{R}$

Suppose $\exists N \in \mathbb{N}$ $\forall n \geq N \Rightarrow$

$$y_n \leq x_n \leq z_n$$

$\Rightarrow \{x_n\}_{n \in \mathbb{N}}$ is convergent and

$$\lim_{n \to \infty} x_n = \bar{x}$$


y_n
$\bar{x}$
z_n