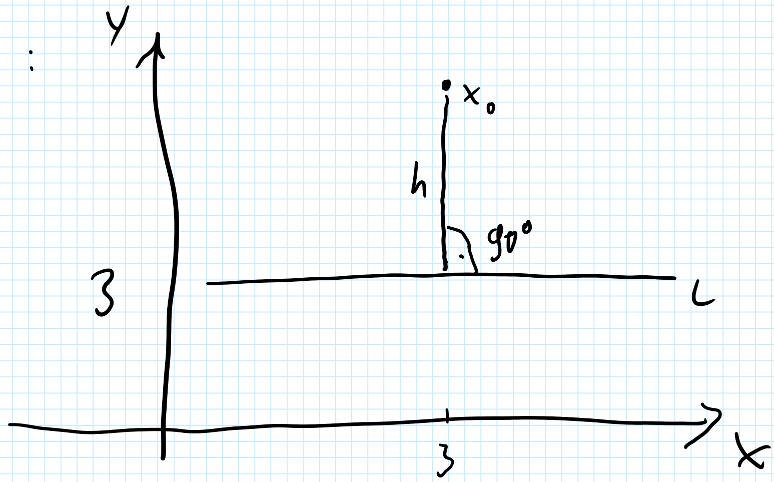
Chapter 1 Vector Spaces

- What is a Vector Space
- Metric + Norm
- Key Properties of Sets
- Limits + Continuity in higher-dimensional Spaces

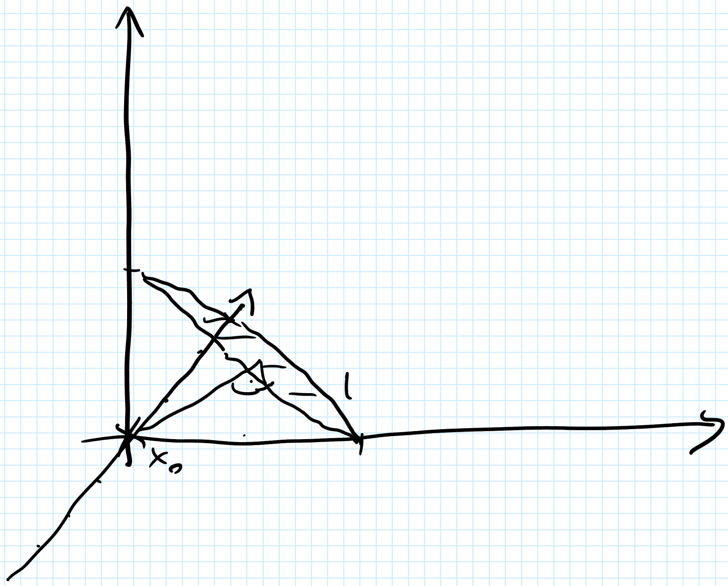
Motivation:



h is orthogonal to l

h gives the shortest distance between x_0 and l

$\mathbb{R}^2$: Graphical intuition



Example 2: for real numbers, we know

$1+2=3$

$2\cdot 2=4$

$\vdots$

What about functions, matrices, vectors?

$$\begin{pmatrix}1\\2\\3\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}a & b\\c & d\end{pmatrix} + \begin{pmatrix}c & a\\d & b\end{pmatrix}$$

$f: \mathbb{R} \to \mathbb{R}$
$g: \mathbb{R} \to \mathbb{R}$, what is $g \cdot f$?

One important concept: Vector Spaces

↳ typically focus on $\mathbb{R}^n$

↳ we can define vector spaces over different sets other than the $\mathbb{R}^n$

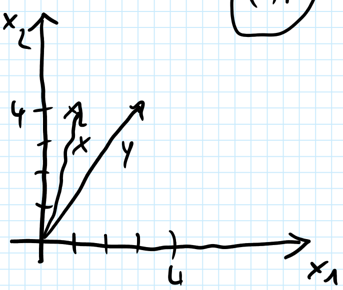
Vectors in $\mathbb{R}^n = \{ (x_1, \dots x_n)' : \forall i \in \{1, \dots, n\} : x_i \in \mathbb{R} \}$

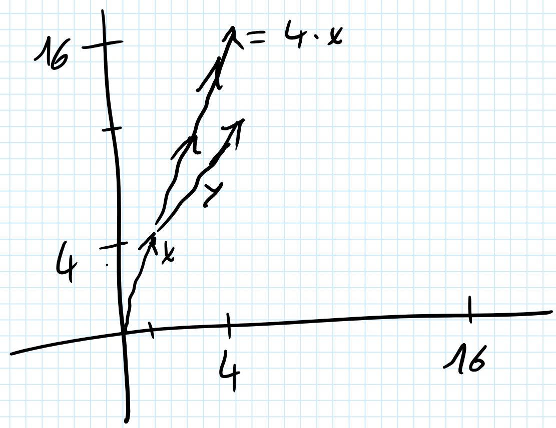
$(x_1, \dots x_n)' = \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$ – column vector

$(x_1, \dots, x_n)$ – row vector

Convention: We usually mean column vectors, unless otherwise stated

Example: $\mathbb{R}^2$ $x = \begin{pmatrix}1\\4\end{pmatrix}$ $y = \begin{pmatrix}3\\4\end{pmatrix}$





Definition: Vector Space

The collection $(V, +, \cdot)$ is called a vector space, if

- V is a non-empty set
- $+$ is the vector addition $+ : V \times V \to V$

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$$(\vec{v}, \vec{w}) \to \vec{v} + \vec{w}$$

- $\cdot$ is the multiplication of vectors with a scalar $\quad \cdot : \mathbb{R} \times V \to V$

$$(a, \vec{v}) \mapsto a \cdot \vec{v}$$

Examples $\quad v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad w = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad a = 5$

$$1 \; \underline{v + w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

$$a \cdot v = 5 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 15 \end{pmatrix}$$

Linear Combinations

Given two vectors $x, y \in V$ and scalars $\lambda, \mu \in \mathbb{R}$ we can combine the two vectors into a new vector by linear combination

$$z = \lambda x + \mu y \in V$$

↳ Span: the set of linear combinations of a set of vectors: span $\subseteq V$

$$\text{span}\left(\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}\right) = \left\{ \begin{pmatrix} \lambda + \mu \\ \mu \\ 0 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_3 = 0 \right\}$$

Basis ~~of a vector space~~: smallest set of vectors to span the space we are interested in

Example: the canonical basis of $\mathbb{R}^3$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = B \qquad \text{Span}(B) = \mathbb{R}^3$$

$\Rightarrow$ Dimension of a vector space is simply the number of elements in the basis

Basis is not unique:

for $\mathbb{R}^3$, $\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}1\\3\\5\end{pmatrix} \right\}$ is this a basis of $\mathbb{R}^3$?

(Exercise: show linear independence)

Linear Independence

consider a set of vectors $S \subseteq V$ and a vector $v \in V$

- v is linearly independent of S if
$$v \notin \text{span}(S)$$
- a set S is linearly independent if $\forall v \in S \quad v \notin \text{span}(S / \{v\})$

Testing linear independence

A set $B = \{b_1, \dots b_k\}$ of vectors is linearly independent if
$$\sum_{j=1}^{k} \lambda_j b_j = 0 \quad \Rightarrow \quad \forall j \in \{1, \dots, k\} \quad \lambda_j = 0$$

Example

$$\lambda_1 \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda_2 \begin{pmatrix}0\\0\\1\end{pmatrix} + \lambda_3 \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}\lambda_1 + \lambda_3\\ 2\lambda_1 \\ 3\lambda_1 + \lambda_2\end{pmatrix}$$

$\lambda_1 + \lambda_3 = 0 \longrightarrow \lambda_3 = 0$

$2\lambda_1 = 0 \Rightarrow \lambda_1 = 0$

$3\lambda_1 + \lambda_2 = 0$

$\hookrightarrow \lambda_2 = 0$

$\Rightarrow \left\{ \begin{pmatrix}1\\2\\3\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \begin{pmatrix}1\\0\\0\end{pmatrix} \right\}$ is linearly independent!

The Scalar Product of Vectors (on the $\mathbb{R}^n$)

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

$$(\vec{x}, \vec{y}) \longmapsto \langle x, y \rangle = x' y = \sum_i x_i \cdot y_i \qquad x = \begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} \qquad x' = (x_1, \dots x_n)$$

- again, the scalar product has to satisfy certain properties
- for simplicity we will look at ... only

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- for simplicity, we will look at examples

Example $x = \begin{pmatrix}1\\2\\3\end{pmatrix} \quad y = \begin{pmatrix}5\\5\\5\end{pmatrix}$

$$\langle x, y \rangle = \underbrace{x' \cdot y}_{\text{Econ}} = (1\ 2\ 3) \cdot \begin{pmatrix}5\\5\\5\end{pmatrix} = 5 + 10 + 15 = 30$$

Also called „dot-product", „inner product", „vector product"

Orthogonality of Vectors

$x, y \in V$ are orthogonal $\Leftrightarrow$ $x'y = 0$

Back to our leading example

- for lines f, g in $\mathbb{R}^n$, this implies that for any two distinct points f_1, f_2 and g_1, g_2 on f, g respectively

$$(f_1 - f_2)'(g_1 - g_2) = 0$$

$$\left(\begin{pmatrix}3\\4\end{pmatrix} - \begin{pmatrix}3\\3\end{pmatrix}\right)' \cdot \left(\begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}3\\3\end{pmatrix}\right)$$

$$\begin{pmatrix}0\\1\end{pmatrix}' \cdot \begin{pmatrix}1\\0\end{pmatrix} = 0 \quad \checkmark$$

$\Rightarrow$ we see:
our formal definition of orthogonality gives the same result as the graphical intuition.

Some practice

① $x = \begin{pmatrix}1\\2\\2\end{pmatrix} \quad y = \begin{pmatrix}3\\0\\2\end{pmatrix} \qquad 2 \cdot x'y$

② Argue why $\forall\, x, y \in \mathbb{R}^n$

- $x'y = y'x$
- $(x+y)' \cdot (x+y) = x'x + 2x'y + y'y$

- $x'x = 0 \Rightarrow x = 0$

③ Show that the canonical basis of the $\mathbb{R}^n$ is linearly independent

$$\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$$

Normed Vector Spaces and Mathematical Distance

What should we expect from a distance measure?

Definition: Metric $\quad X = (X, +, \cdot)$ a real vectorspace. A function $d: X \times X \to \mathbb{R}$ defines a metric on X if

① Non-negativity: $\forall x, y : d(x,y) \geq 0 \wedge d(x,y) = 0 \Leftrightarrow x = y$

② Symmetry " : $d(x,y) = d(y,x)$

③ triangle-inequality $\forall x,y,z \in X \quad d(x,y) \leq d(x,z) + d(z,y)$

Definition: Norm and normed Vector Space

$X = (X, +, \cdot)$ a real vector space

Then $\|\cdot\| X \to \mathbb{R}$ defines a Norm, if

i) $\forall x \in X : \|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$

ii) $\forall x, y \in X \quad \|x+y\| \leq \|x\| + \|y\|$

iii) $\forall x \in X \; \lambda \in \mathbb{R} \quad \|\lambda \cdot x\| = |\lambda| \cdot \|x\|$

If we can define a norm on X, we call X a normed vector space

Examples for norms:

p-norms on $\mathbb{R}^n$ $\quad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$

max-norm on $\mathbb{R}^n$ $\quad \|x\|_\infty = \max_{i \in \{1,\dots,n\}} |x_i|$

For any norm, we can define a norm-induced metric

$$d_N(x,y) = \|x - y\|$$

Why are norm-induced metrics useful?

Why are norm-induced metrics useful?

(1) absolute homogeneity

$\forall x, y \in X \quad \forall \lambda \in \mathbb{R} \quad d_N(\lambda x, \lambda y) = |\lambda| \, d_N(x, y)$

(2) translation invariance

$\forall x, y, z \in X \qquad d_N(x+z, y+z) = d_N(x, y)$

(3) Magnitude of a vector as distance to the origin

$\|x\| = d_N(x, 0)$

<u>Norms are continuous functions</u>

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 : (\|x - y\| < \delta \Rightarrow \big| \|x\| - \|y\| \big| < \varepsilon)$$

Continuity: we can "pull in" limits $\quad \lim_{n \to \infty} \|x_n\| = \|\lim_{n \to \infty} x_n\|$

Example of a <u>very</u> useful metric

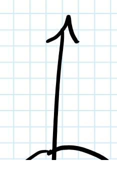
The metric induced by the Euclidean Norm

$$d_N^2(x, y) = \|x - y\|_2 = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2}$$

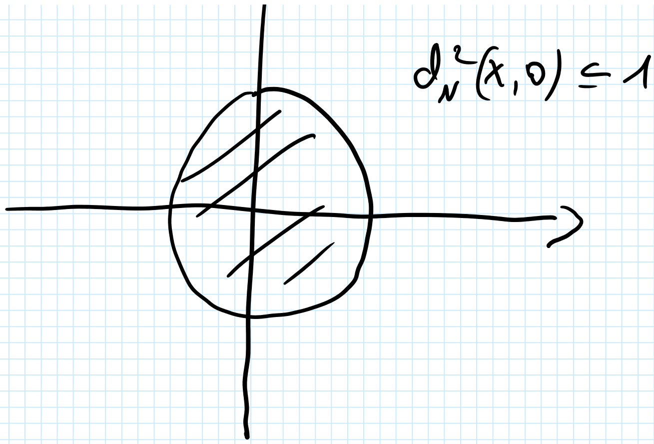
Note that $\quad \underset{y \in \mathbb{R}^n}{\arg\max} \; d_n^2(x, y) = \underset{y \in \mathbb{R}}{\arg\max} \; \underbrace{(d_n^2(x, y))^2}_{\sum_{i=1}^{n} |x_i - y_i|^2}$

$\Rightarrow$ the euclidean norm is closely related to least-squares optimality

Geometric intuition of the Euclidean Norm $(\mathbb{R}^2)$

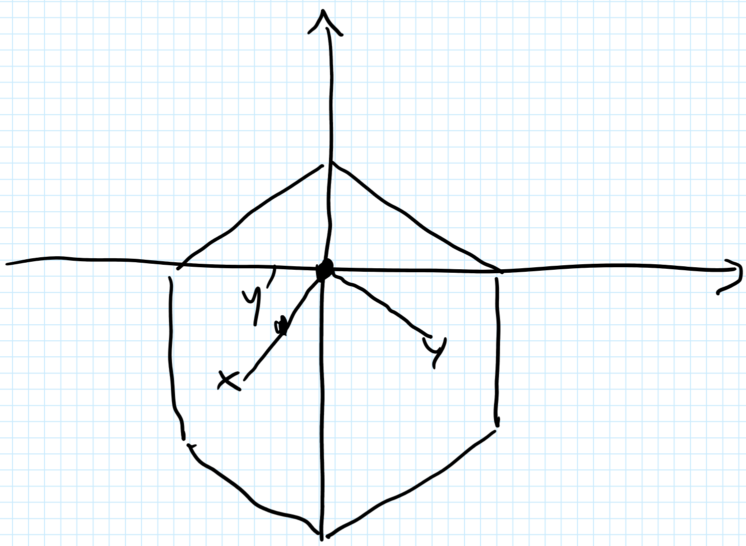


$d_N^2(x, 0) \leq 1$



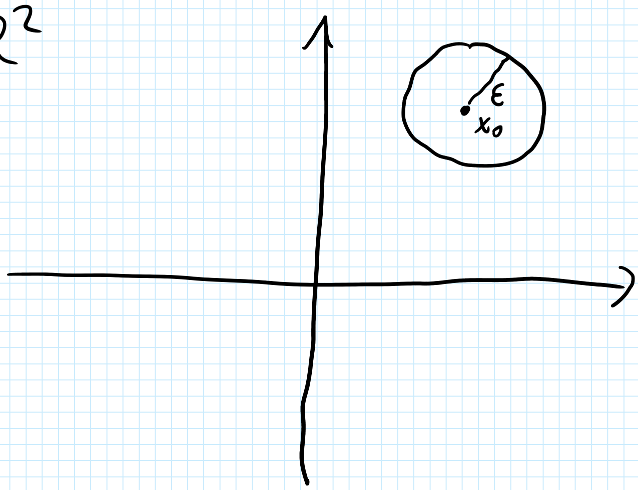
A weird metric: Metric of the french railway

$$d_{FR}(x,y) = \begin{cases} \|x-y\|_2 & \text{if } y = \lambda x \\ \|x\|_2 + \|y\|_2 & \text{else} \end{cases}$$



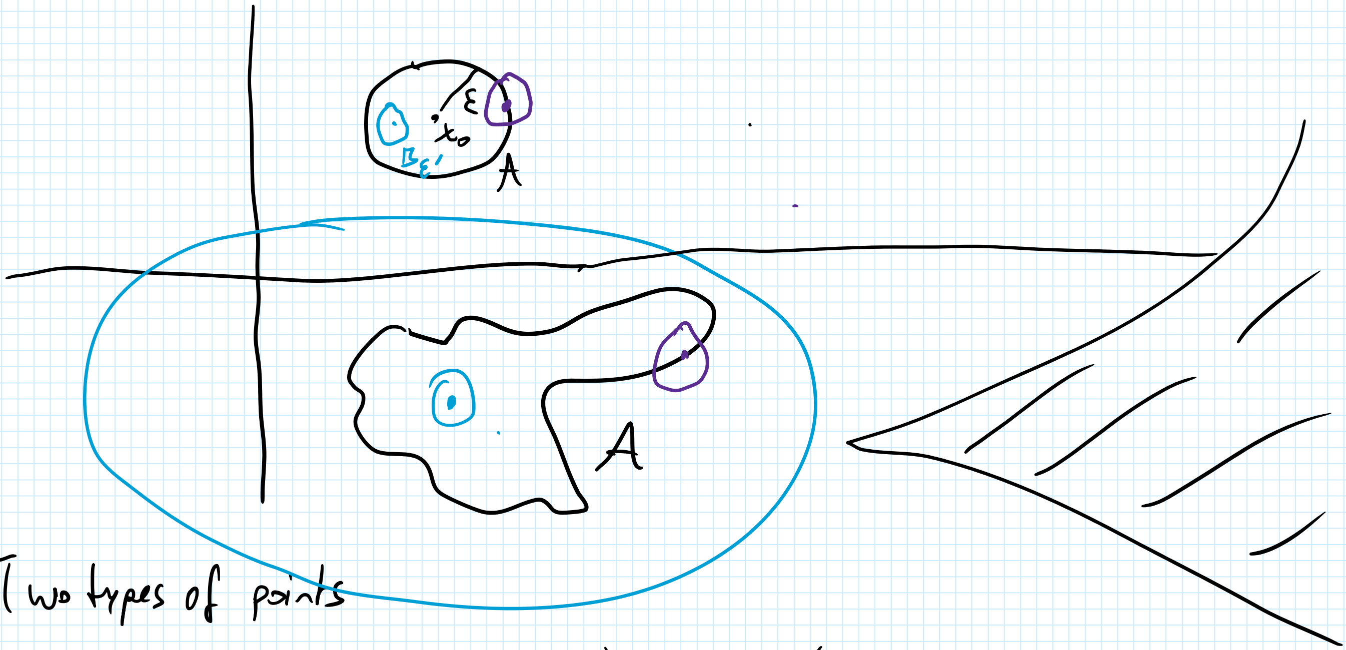
Using Distances for Set Characteristics

$\mathbb{R}^2$



a Ball B

- open Ball : $B_\varepsilon(x_0) = \{ x \in \mathbb{R}^2 : d(x, x_0) < \varepsilon \}$
- closed Ball : $\overline{B}_\varepsilon(x_0) = \{ x \in \mathbb{R}^2 : d(x, x_0) \leq \varepsilon \}$



Two types of points

interior points $x \in int(A) \Leftrightarrow \exists \varepsilon' > 0 \;\; B_{\varepsilon'}(x) \subseteq A$

boundary point $x \in \partial(A) \Leftrightarrow \forall \varepsilon'' > 0 \;\; B_{\varepsilon''}(x) \cap A \neq \emptyset$
$\wedge \; B_{\varepsilon''}(x) / A \neq \emptyset$

<u>Open set</u> : $A = int(A)$ (no boundary points)
ex : $(0,1)$

<u>closed set</u> : $A = int(A) \cup \partial(A)$

<u>Bounded set</u>: the distance of elements in A is smaller than some upper bound
$\exists x \in X \;\; \exists r < \infty \;\; A \subseteq B_r(x) \quad A \subseteq X$

<u>Compact Set</u>: is closed and boundet (not the definition, but the result of a thm (Heine - Borell))

Note : • you can have sets that are neither open nor closed
e.g. $[0,1)$, the half-open interval

• you can have sets that are both open and closed
e.g. $\mathbb{R}$, $\emptyset$

• Why do we care? Compactness is central to optimization.

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Properties of open and closed sets

consider a metric space (X, d)

open
i) $\emptyset$ and X are open in X
ii) $A \subseteq X$ is open $\Leftrightarrow$ $A^c = X/_A$ is closed
iii) for open sets A_i $\bigcup_i A_i$ is open (also for infinite $i = 1, \dots$)
iv) for a finite collection of open sets A_i

$\bigcap_i A_i$ is open

closed
i) $\emptyset$ and X are closed in X
ii) see open ii)
iii) the union of a finite collection of closed sets is closed
$A_i,\ i = 1, \dots n$ $n < \infty$, A_i closed $\forall i \Rightarrow \bigcup_i A_i$ is closed
iv) the intersection of an arbitrary (possibly infinite) collection of closed sets is closed

Thm: Closedness and Sequences

Suppose $(X, +, \cdot)$ is a real vector space and $B \subseteq X$

B is closed $\Leftrightarrow$ for any convergent sequence $\{x_n\}_{n \in \mathbb{N}}$:

$$\forall n \in \mathbb{N}\ x_n \in B \Rightarrow \lim_{n \to \infty} x_n \in B$$

Thm: Weak inequality and the limit

take $(X, +, \cdot)$ a real vector space $f: X \to \mathbb{R}$
$g: X \to \mathbb{R}$

$\forall x \in X$ $f(x) \leq g(x)$

consider $x_0 \in X$

suppose $\exists$ for $g_0 \in \mathbb{R}$: $\lim_{x \to x_0} f(x) = f_0$ and $\lim_{x \to x_0} g(x) = g_0$

$\Rightarrow f_0 \leq g_0$

Thm: Weak inequality and the limit: Sequences

$(X, +, \cdot)$ vector space

consider $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ convergent sequences over X

$\forall n \in \mathbb{N} \quad x_n \leq y_n$ And $\quad x_n\, y_n \in B \; \forall n\in\mathbb{N} \wedge \begin{array}{l} x_n \to x \in B \\ y_n \to y \in B \end{array}$

$\Rightarrow \; x \leq y$

Thm: Checking boundedness

$(\mathbb{X}, d)$ a metric space , d a norm-induced metric

Let $A \subseteq X$: A is bounded if $\|\cdot\|$, the norm that induced d is bounded on A : $\exists\, b < \infty \; (\forall x \in A : \|x\| < b)$

Example:

The budget set $\quad B(y \mid p_1, p_2) := \{x \in \mathbb{R}^2 : p_1 x_1 + p_2 x_2 \leq y\}$

if $p_1, p_2 > 0$, the budget set is closed and bounded!

Convergence of functions

$f : X \to Y$ $\qquad X \subseteq (\mathbb{X}, \|\cdot\|_x)$ subset of a metric vector space

$\qquad Y \subseteq (\mathbb{Y}, \|\cdot\|_y)$ subset of a metric vector space

f_a is the limit of f at a in X if

$$\forall\, \varepsilon > 0 \; \exists\, \delta > 0 : \forall x \in X \left(\|x - a\|_x < \delta \Rightarrow \|f(x) - f_a\|_y < \varepsilon \right)$$

Continuity of a function at x_0

f is continuous at x_0 if $f(x_0) = \lim\limits_{x \to x_0} f(x)$

$x \to x_0$

(you can also use the ε - δ - Notation)

↳ Prove that sth is not continuous at x
find sequence, such that $x_n \xrightarrow{n \to \infty} x$ for which
$f(x_n) \not\to f(x)$

Continuous function: is continuous on all $x \in \text{Domain}(f)$

Convexity of sets

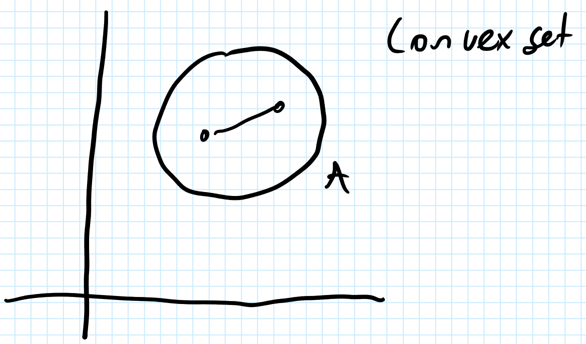
- Economists are not always lucky enough to work with vector spaces
- We can, however, preserve a lot of structure on convex sets

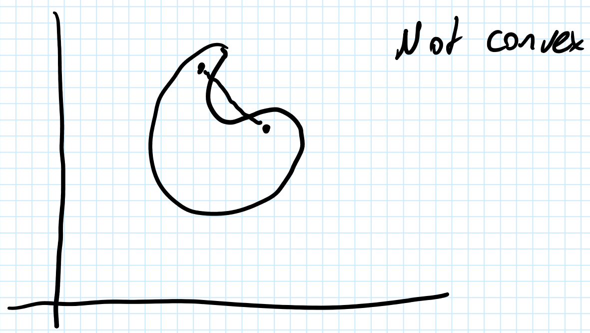
Def: Convex combination

Let $(X, +, \cdot)$ be a real valued vector space
A convex combination ~~of vectors~~ x^c of vectors $x_1, \dots x_n$
is a linear combination $\sum_{i=1}^{n} \lambda_i x_i \quad \forall_i \; \lambda_i \geq 0 \quad \sum_i \lambda_i = 1$

Convex Set:

A set that contains all convex combinations of any two elements
$\forall\, a_1, a_2 \in A \quad \forall \lambda \in [0,1] \quad \lambda a_1 + (1-\lambda) a_2 \in A$





Proposition: Convexity preserving operations

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$(X, +, \cdot)$ real vector space

i) X, $\emptyset$ are convex

ii) $A \subseteq X$ is convex $\quad \alpha A := \{\alpha \cdot a : a \in A\}$ is convex $\forall \alpha \in \mathbb{R}$

iii) $A, B \subseteq X$ convex $\quad A + B = \{ \quad c \quad \{a + b : a \in A, b \in B\}$ is convex

iv) if $\{A_i\}_{i \in I}$ is a (potentially infinite) collection of convex sets

$\bigcap_{i \in I} A_i$ is convex