E600 Mathematics

Fall Semester 2021

Self-study Exercises 3

to be solved by August 29

Topics: Multivariate Differentiation and Integration

Exercise 1: Important Objects of Differential Calculus

Consider $x_0 \in X \subseteq \mathbb{R}^n$, $f : X \mapsto \mathbb{R}$. What is the type (e.g., real number, matrix, function, operator, etc.) of the following objects:

$$\frac{\partial f}{\partial x_1}(x_0), \qquad \frac{df}{dx}, \qquad \frac{\partial}{\partial x_1}, \qquad \nabla f, \qquad \frac{df(x_0)}{dx}$$

What, if any, is the difference between $\frac{df}{dx}(x_0)$ and $\nabla f(x_0)$?

Exercise 2: Invertability

a.) Monotonic Functions and Injectivity (online)

Show that any strictly monotonic function is injective.
Hint: It may be helpful to look up the formal definitions of strict monotonicity and injectivitiy. From there, it should not be a far way to showing this fact.

b.) Some Examples

Determine which of the following functions are invertible, and if not, which criterion (injectivity or surjectivity) fails. In case of non-invertability, can you restrict the domain and/or codomain to achieve invertability?[1]

1. $f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto \cos(x)$
2. $f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto x^2$
3. $f : \mathbb{N} \mapsto \mathbb{N},\ n \mapsto n^4$
4. $f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto \mathbb{1}[x < 1](x-1)^2 + \mathbb{1}[x \geq 1]\log(x)$

[1] You should understand the word “restrict” as cutting the set to the left/right, but do not manipulate it in the middle; e.g. if the initial set is $\mathbb{N}$, then $R = \mathbb{N} \cap [3, 100]$ is an allowed restriction, but $R = \{3, 9, 27, 87\}$ is not.

Hint: for 4., investigate the two parts of the domain, $x < 1$ and $x \geq 1$, separately, and then think about whether putting the two parts together changes your conclusion. It may also be helpful to draw the function.

c.) Monotonic Functions and Injectivity: Application (online)

Consider the function $f : \mathbb{R} \mapsto \mathbb{R}, x \mapsto \sin(x) - \frac{3}{2}x$. Is this function injective?
Comment: We will investigate surjectivity in the in-class exercises for this function.

Exercise 3: Convexity

a.) Set Convexity: revisited (online)

Is the following set convex? Justify your answer!

$$S := B_\varepsilon(x_0) = \{x \in X : \|x - x_0\| < \varepsilon\}$$

for some $\varepsilon > 0$. What about a closed ball?

b.) Norms (online)

Investigate the following function with respect to (strict) convexity/concavity:

$$f : \mathbb{R}^n \mapsto \mathbb{R}, x \mapsto \|x\|$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$, $n \in \mathbb{N}$.
Hint 1: We know that norms are continuous, but they need not be differentiable. Hence, the criterion for the second derivative (that you may know from the lectures, if we managed to cover it already) is not useful here, and it is instructive to proceed with the "raw" definition of convexity.
Hint 2: The solution of a.) offers some insight into how we should approach this problem.
Hint 3: A function is only both convex and concave if it is linear, i.e. $f(x + y) = f(x) + f(y)$ for *any* possible arguments x, y. Thus, once you showed that f satisfies one of the properties, you can check the other by investigating whether f is/can potentially be linear.

Exercise 4: Multivariate Differentiation

a.) Taylor Approximation: Univariate case (online; slightly modified)

This exercise is meant as a means to get started with Taylor's theorem for those who did not cover it in their undergraduate or need to fresh up their memory. This less notation-intense context should help you deal with multivariate Taylor approximations as used in the next exercise. Feel free to skip ahead to *(ii)* if you are already well-familiar with Taylor approximations.

1.: First and Second Order Approximation. Compute the first and second order Taylor approximations to the exponential function $\exp : \mathbb{R} \mapsto \mathbb{R}_+, x \mapsto \exp(x)$ around $x_{0,1} = 1$. Evaluate both approximations at $x = -5$ and $x = 2$, and compare the approximation quality. Is one always better than the other? Are the approximations "good", i.e. are they close to the true value?

2.: **Higher Order Approximation.** Compute the n-th order Taylor Approximation to the exponential function for $x_0 = 0$ for variable $n \in \mathbb{N}$. Can you find an infinite sum representation for the exponential function using polynomial terms?

b.) Matrix Functions (online)

Consider a matrix $A \in \mathbb{R}^{n \times n}$, $n \in \mathbb{N}$.

(i) Show that $\frac{d}{dx}(Ax) = A$.

(ii) What is the derivative of $f : \mathbb{R}^n \mapsto \mathbb{R}, x \mapsto x'Ax$?
Hint: Use *(i)* and the multivariate product rule.

(iii) If $A = \begin{pmatrix} 1 & \alpha \\ \beta & 4 \end{pmatrix}$, can you find values for α and β so that the second derivative of $x'Ax$ is positive definite everywhere? Can you find an alternative combination where A is positive semi-definite but not positive definite?

Exercise 5: Multivariate Differentiation

a.) Hessian Criterion for Convexity (online)

Investigate the following function with respect to (strict) convexity/concavity:

$$f : \mathbb{R}^2 \mapsto \mathbb{R}, x = (x_1, x_2)' \mapsto \exp(x_1) + x_1 x_2 + 5x_1 + 4$$

Hint: Recall that we can use the second derivative to investigate convexity. The function is infinitely many times continuously partially differentiable, which can save you a few computational steps.

b.) Support Restriction?

Use the Hessian Criterion to investigate whether the function

$$f(x_1, x_2) = \frac{1}{2}(x_2^3 + 2x_1 x_2 + x_1^2)$$

is (strictly) convex or concave on $\mathbb{R}^3$, and otherwise try to find the support restrictions on which one of the properties holds.
Hint: The function is infinitely many times continuously partially differentiable, which can save you a few computational steps.

c.) Multivariate Taylor and Cobb-Douglas

For what follows, consider a household with Cobb-Douglas utility over consumption c and leisure l, i.e. $u(c,l) = c^\alpha l^{1-\alpha}$ with $\alpha \in (0,1)$. You can use that at points $(c,l) \neq (0,0)$, this function is infinitely many times differentiable.

1.: Approximation of Order 1. Compute the Taylor approximation of order 1 to $u(c,l)$ at $(c_0,l_0) = (1,1)$. For $\alpha = 1/2$, compare the approximated values for $(c,l) = (3/2,1/2)$ and $(c,l) = (5,4/5)$ to the true value of u.

2.: Approximation of Order 2. Compute the Taylor approximation of order 2 to $u(c,l)$ at $(c_0,l_0) = (1,1)$. For $\alpha = 1/2$, compare the approximated values for $(c,l) = (3/2,1/2)$ and $(c,l) = (5,4/5)$ to the true value of u.

Write down the first order Taylor expansion. You may use $\lambda \in (0,1)$ as an unknown variable.

Exercise 6: Multivariate Integration

Consider an economy populated by a mass $[0,1]$ of firms that use capital k and labor l to produce output $y = f(k,l) = Ak^\alpha l^{1-\alpha}$, i.e. they use the same Cobb-Douglas production technology. Further, suppose that economy-wide output satisfies

$$Y = \int_{[0,1]\times[0,1]} f(k,l)d(k,l).$$

Amongst others, this relationship can be obtained from assuming that labor l and capital k are independently and uniformly distributed on $[0,1]$. However, it is not too important what this means here, it just ensures that the equality above holds.

Determine Y as a function of A and α. What do you conclude for the role of α, the relative importance of capital in the production process in terms of its relationship to Y?