# E600 Mathematics

Fall Semester 2023

# Self-study Exercises 1

to be solved by August 25

*Topics: Vector Space, Metric and Norm, Set Properties*

## Exercise 1: Key Concepts of Vector Spaces

### a.) The Scalar Product (online)

Compute the scalar product of $v$ and $w$ when...

1. $v = (2,3)'$ and $w = (-2,9)'$
2. $v = (3,2,\ln(8),-6)'$ and $w = (\ln(2),1,-1,1/4)'$ (*Hint:* $a\ln(b) = \ln(b^a)$)
3. $v = (4,19)'$ and $w = (2,3,5)'$

### b.) More Scalar Products

Compute $a \cdot b$ for

1. $a = (2,5,1)'$, $b = (1,1,3)' \in \mathbb{R}^3$
2. $a = (2,0,-3,4)$, $b = (9,-8,7,-6) \in \mathbb{R}^4$

What do you tell your colleague who claims to have found $v \in \mathbb{R}^n$ so that $v \cdot v = -1$?

### c.) The Binary Metric

Consider a real vector space $\mathbb{X} = (X,+,\cdot)$, and define the binary metric

$$d_B : X \times X \mapsto \mathbb{R}, d_B(x,y) = \mathbb{1}[x \neq y]$$

Show that the function indeed constitutes a metric, i.e. show that it satisfies the three properties that define a metric function.

In case you are not familiar with the notation used here, $\mathbb{1}[S(x)]$ is a so-called indicator function for a statement $S(x)$ related to $x$ that takes the value 1 when $S(x)$ is true and 0 otherwise. Accordingly,

$$d_B(x,y) = \mathbb{1}[x \neq y] = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

### d.) A Norm (online)

The course discusses the most common examples of norm functions as used in practice. However, there are of course a variety of other functions that are norms. One such example is the following function on $\mathbb{R}^3$:

$$n : \mathbb{R}^3 \mapsto \mathbb{R}, x = (x_1, x_2, x_3)' \mapsto |x_1| + \max\{2|x_2|, 3|x_3|\}.$$

Verify that this function constitutes a norm according to our definition by establishing the three norm properties for $n$.

### e.) Norm Equivalence (online)

A key concept related to norms that the course does not discuss is *norm equivalence*. You may recall that we said that a function may be continuous in one metric space but not in the other, depending on the metric chosen. Similarly, convergence of sequences may depend on the metric chosen. A further appealing aspect of norm-induced metrics over general metrics is that at least in metric spaces over $\mathbb{R}^n$ with finite $n \in \mathbb{N}$, all of our usual p-norms are *equivalent*, which means that if a function $f$ satisfies continuity or a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges with respect to one norm, they do so with respect to the other as well.

It turns out to be sufficient for norm equivalence that we can reduce the differences in two norms to some constants that matter little in the $\varepsilon/\delta$-arguments we use to investigate convergence, continuity and related concepts. Formally, two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on $X$ are said to be equivalent if there exist constants $0 < c \leq C < \infty$ such that

$$\forall x \in X : c\|x\|_a \leq \|x\|_b \leq C\|x\|_a.$$

Show that all p-norms are equivalent on any $\mathbb{R}^n$, i.e. that $\forall p, q \in \mathbb{N} \cup \{\infty\}$, $\|\cdot\|_p$ is equivalent to $\|\cdot\|_q$ on $\mathbb{R}^n$ for any $n \in \mathbb{N}$. (*Hint:* this is easiest done in two steps: (1) show that any $p$-norm is equivalent to the maximum norm, and (2) show that if a $p$- and $q$-norm are both equivalent to the maximum norm, the $p$- and $q$-norm are also equivalent to each other.)

## Exercise 2: Testing for Set Properties

In this exercise, we practice the investigation of key set properties discussed in lecture 1. As set properties refer to metric or normed vector spaces, respectively, you need to know which distance measure to consider. You can assume (as always when nothing else is explicitly specified) that we deal with Euclidean spaces, but feel free to choose any other $p$-norm (including

$p = \infty$) that may work better for you (recall the previous exercise: $p$-norms are equivalent, it does not matter which one you choose).

*Remark:* The numbering of sets is intentional; $S_2$ was skipped on purpose so as to keep the notation of the online solutions.

### a.) A subset of the real line (online)

Let $S_1 := \{x \in \mathbb{R} : x^2 \leq 4\}$. Is this set open, closed, compact and/or convex?

### b.) Two dimensions (online)

Let $S_3 := \{x \in \mathbb{R}^2 : x_1 \leq 3\}$. Is this set open, closed, compact and/or convex?

## Exercise 3: Testing for Set Properties, again (online)

Let $S_4 := \{x \in \mathbb{R}^2 : x_1 x_2 = 5\}$. Is this set open, closed, compact and/or convex?
*Hint:* $f : \mathbb{R}^2 \mapsto \mathbb{R}, x = (x_1, x_2)' \mapsto x_1 x_2$ is continuous, and thus $\lim_{n\to\infty} x_1^n x_2^n = \lim_{n\to\infty} x_1^n \lim_{n\to\infty} x_2^n$.

## Exercise 4: Basis of a Vector Space

### a.) Bases of the $\mathbb{R}^2$

Which of the following sets are bases of the $\mathbb{R}^2$ (S&B, Ex. 11.12)?

$$S_1 = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-2\\-2\end{pmatrix}\right\}, \quad S_2 = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\-2\end{pmatrix}\right\}, \quad S_3 = \left\{\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-2\\2\end{pmatrix}\right\}, \quad S_4 = \left\{\begin{pmatrix}1\\-1\end{pmatrix}, e_2\right\}$$

### b.) A Basis of a Function Space

Consider the set of second order polynomials,

$$\mathbb{P}_2(X) := \{f : X \mapsto \mathbb{R} : (\exists a, b, c \in \mathbb{R} : f(x) = ax^2 + bx + c)\}$$

It turns out to be the case that $\mathbb{P}_2(X)$ is a subspace of $\mathbb{F}(X, \mathbb{R})$, the space of all functions mapping from $X$ to $\mathbb{R}$. Find a basis for $\mathbb{P}_2(X)$, i.e. a set of functions $f_1, f_2, \ldots, f_k$ such that (1) every second order polynomial can be written as a linear combination of these functions and (2) the functions are all linearly independent, i.e. they can not be written as linear combinations of each other.

*Bonus Question:* Show that $\mathbb{P}_2(X)$ is a subspace of $\mathbb{F}(X, \mathbb{R})$. What is its dimension?