E600 Mathematics

Fall Semester 2023

Self-study Exercises 0

to be solved by August 23

Topics: Mathematical Fundamentals

This is the first set of self-study exercises. These exercises are meant to be worked on by you, individually or in groups. Solving the problems before we discuss them in class is highly recommended as a great deal of mathematical knowledge comes only through application and practice. However, there is no need to hand in solutions, and also no punishment if you don't find the time to work on them.

Some exercises are marked with "online". This means that they are taken from the respective chapter's collection of exercises on the webpage of the course (`e600.uni-mannheim.de`), where you can also find the solutions. You should try to solve the problems without looking at the solution, but for these exercises, you can check if your approach works or what you're missing, which will help for the remaining problems.

The solutions are discussed in class – if we are short on time, some online problems might be left out. That being said, your first goal in working on these exercises is understanding what the problem is and developing a rough idea of how to approach it. Doing so will already help a lot as this will make our discussions of the solutions much more accessible.

Exercises for Chapter 0

Exercise 1: Notation and Logic

a.) Validity of Arguments

Consider an argument with structure "Premise 1 and Premise 2 imply Conclusion". Is the argument valid for the given combinations of premises and conclusion?
Hint 1: Validity is given only if the premises *necessarily* imply the conclusion, it does not suffice if the premises do not contradict the conclusion.
Hint 2: Recall that you can use circles to illustrate mathematical arguments.

Nr.	Premise 1	Premise 2	Conclusion
1	All dogs do not meow	Snoopy is a dog	Snoopy does not meow
2	All cats dislike rain	Snoopy dislikes rain	Snoopy is a cat
3	A free person has nothing to lose	A prisoner is not a free person	A prisoner has something to lose
4	If it rains, we don't play outside	We play outside	It's not raining
5	For all $x \in S$, if $x > 1$, then $x > 2$	$0 \in S$ and $0 > 1$	$0 > 2$

Is 5. sound if $S \subseteq \mathbb{R}$, and the "$>$" relation is defined in the usual way?

b.) Arguments and Sets

1. Define some appropriate notation and write down Nr. 1 of 1.a) as a set statement.
2. Draw Nr. 2 of 1.a.) using the "circle approach" to sets.

c.) Quantifiers and Implication (online)

Assess whether the following statements are necessary, sufficient, equivalent, or neither of the previous, for $S := (\forall x \in A : (x - \pi \in \mathbb{Z}))$. You may assume that A is not the empty set, so that it contains at least one element.

1. $\exists x \in A : (x - \pi \in \mathbb{Z})$
2. $\forall x \in A : (x - \pi \in \mathbb{N})$
3. $\exists! x \in A : (x - \pi \in \mathbb{Z})$
4. $\nexists x \in A : (x - \pi \notin \mathbb{Z})$
5. $\nexists x \in A : (x - \pi \notin \{1, 2, 3\})$
6. $A = \{1, 2, 3\}$
7. $A = \{1 + \pi, -1 + \pi\}$
8. $\forall x \in A : (x - 4 \geq 2)$

Exercise 2: Set Theory

a.) Set Operations (online)

Compute union, intersection and both set differences for $A = \{1, 3, 5, 7, 9\}$ and $B = \{-1, 0, 1, 2, 3, 4, 5\}$.

b.) Statements related to Sets

Let $A = \{2,4,6,8,10\}$ and $B = \{1,3,5,7,9\}$. Which of the following statements are true?

1. $2 \in A$
2. $3 \ni B$
3. $4 \notin B$
4. $A \in \mathbb{N}$
5. $A = \{2n : n \in \mathbb{N}\backslash\{0\}\}$
6. $A \cup B = \mathbb{N}$
7. $A \cup B \subset \mathbb{N}$
8. $A = \{2,4,6,8,10,2,4,6,8,10\}$
9. $A = \{2,4,6,8,10,\{2,4,6,8,10\}\}$
10. $B = \{n \in \mathbb{N} : ((\exists m \in \mathbb{N} : n = 2m+1) \vee n < 10)\}$
11. $B = \{n \in \mathbb{N} : ((\exists m \in \mathbb{N} : n = 2m+1) \wedge n < 10)\}$
12. $A = [2,10) \cap \mathbb{N}$

For the first and last statement, if they are false, can you modify them to make them true?

Exercise 3: Functions

a.) Codomain and Range

Give an example for a function f for which the codomain is not equal to the range of f.

b.) Image of a Set under a Function

Let $f : X \to Y$ be a function, and let $A \subset X$. If we say that y is an element of $f[A]$, i.e. $y \in f[A]$ what exactly do we know about y?
A. $f(y) \in A$.
B. $f^{-1}(y) \in A$.
C. $y \in X$.
D. For some $x \in A$, it holds that $f(x) = y$.
E. $y \in A$.

c.) Preimage of a Set under a Function

Let $f : X \to Y$ be a function, and let $B \subset Y$. If we say that x is an element of $f^{-1}[B]$, i.e. $x \in f^{-1}[B]$, what exactly do we know about f and x?
A. $f(x) \in B$
B. $\exists y \in B : x = f^{-1}(y)$
C. $x \in B$
D. $f(x) = B$
E. f is invertible.
F. $f^{-1}(B) = x$

d.) Derivative using the Appropriate Rule

Calculate $f'(x)$ for $f : \mathbb{R} \mapsto \mathbb{R}, x \mapsto \sin((2x+4)^2)$.