E600 Mathematics

Fall Semester 2021

In-class Exercises

To be solved in class on Wednesday and Friday, week 2

Exercises for Chapter 0

Problem 1: Notation and Logic

a.) Writing Formal Statements (online)

Write down the following verbal statements in formal notation.
Example. All real numbers are also rational numbers.
Answer: $\forall x \in \mathbb{R} : (x \in \mathbb{Q})$

1. The set A contains the number 5.
2. The set B contains the number 5 but not the number 4.
3. No natural number is strictly negative (that is, strictly smaller than zero).
4. If x is positive and y is negative, then (this implies that) the product $x \cdot y$ is negative.
5. At any multiple of π, the sin-function is equal to zero.
6. For any natural number and any integer, if the integer is positive, then their product is positive.

b.) Negation of Statements

Negate the following statements. Is the negation true (1.-3.)?

1. $\exists n \in \mathbb{N} : n < 0$.
2. $\forall x \in \mathbb{R} : (x - 1 > 0 \Rightarrow x > 0)$.
3. $\forall x \in \mathbb{R} : x \in \mathbb{N}$.
4. $P \lor Q$, where P and Q are arbitrary statements.

Problem 2: Set Theory

a.) Cardinality and Power Set

Remember that the cardinalty of a set A, $|A|$, denotes the number of elements in a set.

1. What is the value of $|\emptyset|$?
2. What is the value of $|\{\emptyset\}|$?
3. If $A = \{1, \pi\}$, what is the value of $|\mathcal{P}(A)|$?
4. Express the value of $|\mathcal{P}(A)|$ as a function of $|A|$.

b.) Relationship of Set Operations (online)

Consider the relationship $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$. Is it true for arbitrary sets A, B and C? Can you give a reasoning for why or why not, respectively? What about the analogous statement when we change intersections to unions and vice versa?

Problem 3: Functions

a.) A Derivative (online)

Take the derivative of $f : \mathbb{R} \mapsto \mathbb{R}, x \mapsto f(x) = \cos(x)/x^2$.

b.) A Limit (online)

Compute $\lim_{x \to 0} \frac{x}{e^x - 1}$. (Hint: L'Hôpital's rule)

Exercises for Chapter 1

Problem 1: Subspaces, Linear Dependence and Basis

a.) A Proper Subspace of $\mathbb{R}^3$

Prove that

$$S_2 := \{x = (x_1, x_2, x_3)' \in \mathbb{R}^3 : x_2 = 0\}$$

gives rise to a proper subspace of $\mathbb{R}^3$. What is its dimension?
Hint: use the linear combination definition of the subspace.

b.) Bases

Think of two different bases for $\mathbb{R}^3$. Include $b_1 = (1,1,0)'$ and $b_2 = (1,0,4)$ in the second.

Problem 2: Norm and Metric in Vector Spaces

a.) The Norms we use are actually Norms

Recall the most commonly used norms on $\mathbb{R}^2$:

- 1-norm ("Manhattan"): $\|x\|_1 = |x_1| + |x_2|$
- 2-norm ("Euclidean"): $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$
- infinity-norm ("Maximum"): $\|x\|_\infty = \max\{|x_1|, |x_2|\}$

(i) Show that the Euclidean norm constitutes a norm.
Hint: You may use the **Cauchy-Schwarz inequality** for the Euclidean space $(\mathbb{R}^n, \|\cdot\|_2)$, which states that for any $x, y \in \mathbb{R}^n$:

$$|x \cdot y| \leq \|x\|_2 \|y\|_2.$$

(ii) Except for the triangle inequality, the norm property proofs for the other norms are highly analogous. To convince yourself that the Maximum norm is also a norm, show that it satisfies the triangle inequality.

(iii) Sketch the unit-closed ball of these norms, i.e. $\bar{B}_1(0) = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$

Remark: The arguments establishing that the norms we considered here constitute norms on any $\mathbb{R}^n$, $n \in \mathbb{N}$ proceed in perfect analogy to the solutions of this exercise.

b.) Inverse Triangle Inequality

Let $(\mathbb{X}, \|\cdot\|)$ be a normed vector space. Show the inverse triangle inequality, that is, prove that

$$\forall x, y \in \mathbb{X} : \|x - y\| \geq \big|\|x\| - \|y\|\big|.$$

c.) Norm Continuity

Show that any norm is continuous. More formally: show that if $\mathbb{X} = (X, +, \cdot)$ is a vector space and $\|\cdot\|$ defines a norm on X, then it holds that for any $x_0 \in X$,

$$\forall \varepsilon > 0 \exists \delta > 0 : (x \in B_\delta(x_0) \Rightarrow \|x\| \in B_\varepsilon(\|x_0\|))$$

or equivalently

$$\forall \varepsilon > 0 \exists \delta > 0 : (\|x - y\| < \delta \Rightarrow \big|\|x\| - \|y\|\big| < \varepsilon).$$

Exercises for Chapter 2

Problem 1: Laplace Expansion (online)

In the course, we said that we typically deal with matrices of manageable size when computing the determinant, or that the matrix has a convenient structure (triangular, diagonal), where computing the determinant is as simple as multiplying all diagonal elements to obtain the trace. In some applications, however, you may not be that lucky, and revert to the general Laplace rule. Especially if there are zeros in the matrix, this method is still quite easily and quickly applied; this exercise is supposed to convince you of this fact.

Compute $\det(A)$ when

$$A = \begin{pmatrix} 1 & 3 & 0 & 0 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ 0 & 12 \cdot \pi & 3 & -5 & 1 \\ 0 & 2 & 0 & -1 & 1 \\ 1 & 1 & 0 & 2 & 3 \end{pmatrix}$$

Problem 2: The Nullspace and the Dimension of the Solution Set

A key concept related to solving equation systems in matrix notation that we haven't touched in the lecture is the "Nullspace" of a matrix A, also called the kernel, defined as

$$\ker A = \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}.$$

It is straightforward to verify the subspace property since $A(\lambda x + \mu y) = \lambda Ax + \mu Ay$. Here, we deal with its relation to the set of solutions. It will allow us to more formally address our intuition of free variables through the fundamental theorem of Linear Algebra, a really powerful result that you should have seen at least once!

a.) Solutions and the Kernel

Show that if there exists a solution x^* to the equation system $Ax = b$ in matrix notation, then x^s is a solution if and only if there exists an $x_0 \in \ker A$ so that $x^s = x^* + x_0$.

Also answer the following:

1. What can you conclude for the dimension of the number of free dimensions in the problem?

2. Suppose that $B_K(A) = \{b_{K,1}, \dots, b_{K,d}\}$, $d \in \{0, 1, \dots, m\}$, is a basis of $\ker A$. How can you use $B_K(A)$ to represent the solutions of $Ax = b$?

b.) The Fundamental Theorem of Linear Algebra

The theorem tells us about the interrelation of the number of free dimensions and the rank: it states that for $A \in \mathbb{R}^{n \times m}$,

$$\dim(\ker A) = m - \text{rk } A.$$

Note that unique solutions to $Ax = b$ can only exist if $\dim(\ker A) = 0$ (recall a.)), which already rules out unique solutions if $n < m$, i.e. strictly more unknowns than equations.

Noting that $(1,1,1)'$ is a solution, apply the theorem to determine the number of free variables in $Ax = b$ when

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

and use a.) to characterize the set of solutions.

Exercises for Chapter 3

Problem 1: Intermediate Value Theorem (online)

Here, we consider a new theorem that the lecture had not introduced:

Theorem 1. ***(Intermediate Value Theorem)*** *Let $f : X \mapsto \mathbb{R}$ for some set $X \subseteq \mathbb{R}$, and assume that f is continuous. Then, for any $a, b \in X$ with $a < b$ and $f(a) \leq f(b)$ (and $f(a) \geq f(b)$), for any $y \in [f(a), f(b)]$ (for any $y \in [f(b), f(a)]$), there exists $c \in [a, b]$ with $f(c) = y$.*

Verbally, this theorem relates to the intuition of being able to draw continuous functions without lifting the pen: if the continuous function attains two different values within the codomain, it will also reach every value in between along the way. The exercise to follow extends this intuition by establishing that for two continuous functions, if one lies above the other at one point but below at another point, then the functions must intersect in between the points.

a.) Intersecting Continuous Functions

Use the intermediate value theorem to show that if two continuous functions f, g with domain $X \subseteq \mathbb{R}$ and codomain $\mathbb{R}$ are such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$ for some $a, b \in X$, then there exists a value $x \in X$ in between a and b (i.e., $x \in [a, b]$ when $a \leq b$ and $x \in [b, a]$ else) such that $f(x) = g(x)$.

b.) Surjectivity

Consider the function $f : \mathbb{R} \mapsto \mathbb{R}, x \mapsto \sin(x) - \frac{3}{2}x$. Is this function surjective? Is it invertible? *Hint:* Use the result of a.). We know about injectivity from the self-study exercises.

Problem 2: Mean Value Theorem

Prove the Mean Value Theorem, i.e. show that for $f \in D^1(X)$, $X \subseteq \mathbb{R}$, for any $a, b \in X$ so that $a < b$, there exists $x_0 \in (a, b)$ so that

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

What does this imply for the existence of "critical values" of f on X, i.e. points $x \in X$ where $f'(x) = 0$? Illustrate this idea graphically.

Problem 3: Multivariate Chain Rule

To be skipped if we are short on time.

a.) Deriving a more familiar Expression

Let $f : Y \times Z \mapsto \mathbb{R}$, $X = Y \times Z \subseteq \mathbb{R}^n$, i.e. consider a function f of the form $f(x) = f(y,z)$ where y and z are potentially vectors. Further, define $g(z) = f(y(z),z)$, so that we vary y in a specific fashion related to z. Using the multivariate chain rule, derive that

$$\frac{dg}{dz} = \frac{\partial f}{\partial y}\frac{dy}{dz} + \frac{\partial f}{\partial z}$$

or respectively, that for any $z \in Z$,

$$\frac{dg}{dz}(z) = \frac{\partial f}{\partial y}(y(z),z)\frac{dy}{dz}(z) + \frac{\partial f}{\partial z}(y(z),z)$$

b.) Application

Use either version of the multivariate chain rule to derive the marginal indirect utility of consumption for x_2 when $u(x_1,x_2) = \sqrt{x_1 x_2}$ and the budget constraint is $x_1 + 2x_2 = 9$.

Exercises for Chapter 4

Problem 1: Solution Existence for Univariate Concave Functions

Consider a univariate, real-valued function $f : (\underline{x}, \bar{x}) \mapsto \mathbb{R}$, $\underline{x}, \bar{x} \in \mathbb{R}$ so that $\underline{x} < \bar{x}$. Suppose that
(i) f is once differentiable,
(ii) f is concave, and that
(iii) there exist $a, b, c \in (\underline{x}, \bar{x})$ with $a < b < c$ so that $f(a) < f(c) < f(b)$.
Can you argue that f assumes a global maximum on $(\underline{x}, \bar{x})$?

Hint 1. Recall that concavity is a desirable feature in unconstrained maximization.
Hint 2. Think about combining the Mean Value and Intermediate Value Theorem.

Problem 2: Unconstr. Optimization and Matrix Definiteness (online)

In the exercises of Chapter 3, we investigated definiteness of the second derivative of $x'Ax$ for $A = \begin{pmatrix} 1 & \alpha \\ \beta & 4 \end{pmatrix}$. Recall that the second derivative was $A + A'$ and that for $v \in \mathbb{R}^2$,

$$v'(A + A')v = 2(v_1^2 + (\alpha + \beta)v_1 v_2 + 4v_2^2)$$

(i) Can you use an optimization problem approach to find the values for α and β where $A + A'$ is not positive semi-definite, i.e. where there exist $v \in \mathbb{R}^2$ for which $v'(A + A')v < 0$?

To help you with the solution, note that if we write $v = \lambda v_0$, we have

$$v'(A + A')v = 2\lambda^2(v_{0,1}^2 + (\alpha + \beta)v_{0,1} v_{0,2} + 4v_{0,2}^2)$$

Hence, the magnitude does not matter for the property of being strictly negative, and you can reduce the search for v with $v'(A + A')v < 0$ to a direction vector. Because the magnitude of the direction vector does not matter and as if $v_1 = 0$, then the expression is weakly positive, there is no loss in setting $v_{0,1} = 1$ and restricting the search to the value $v_{0,2}$ that yields $v'(A + A')v < 0$.

Problem 3: Saving Time in Optimization

Solve

$$\max \frac{4}{3}x^2 + y + xz \quad \text{s.t.} \quad \|(x, y, z)\|_2 \leq 1$$

where $\|\cdot\|_2$ is the Euclidean norm of the $\mathbb{R}^3$.

You will get a multitude solutions to the FOC. For any one of these, using the second order condition, you would have to check two determinants of the Bordered Hessian, one for a 3×3 and one for a 4×4 matrix. What may help to avoid this is that you can make an argument for solution existence, and also the Lagrangian multiplier condition.

Three simplifications and intuitions can be extremely helpful:

- If $\|(x, y, z)\|_2 < 1$, we can increase the objective varying marginally y.

- By norm non-negativity, $\|(x,y,z)\|_2 \leq 1$ is equivalent to $\|(x,y,z)\|_2^2 \leq 1^2$.
- Closed balls of the $\mathbb{R}^n$ are compact.