Exercises Ch. 4 - part 2

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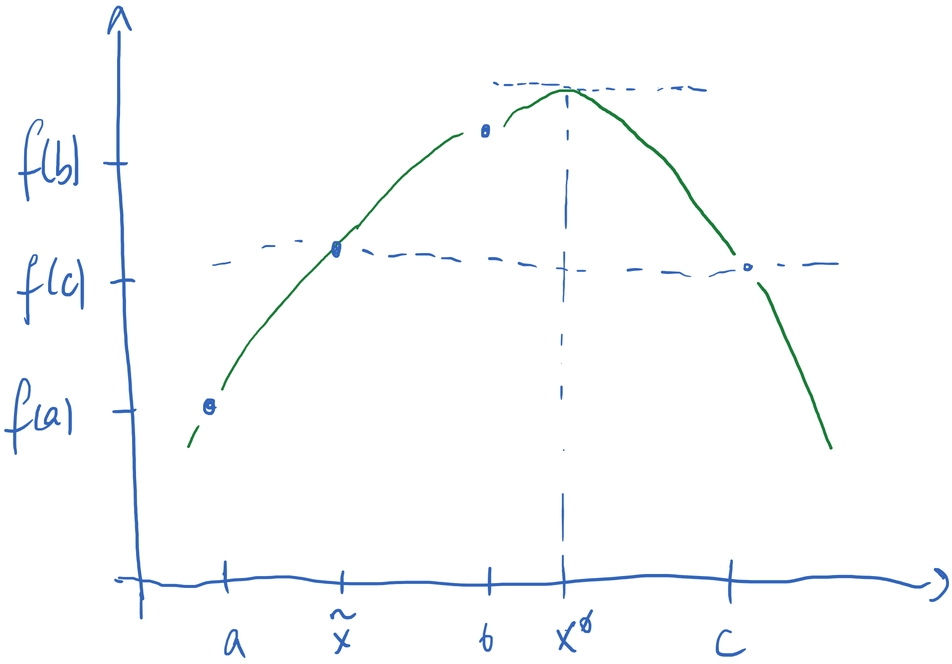
Note: these exercises were not discussed in class.

Exercise IV.3



by mean value thm.:

$\exists \tilde{x} \in (a,b)$:
$f(\tilde{x}) = f(c)$

by the intermediate value thm:

$$\exists x^\emptyset \in [\tilde{x}, c] : \quad f'(x^\emptyset) = \frac{f(c) - f(\tilde{x})}{c - \tilde{x}} = 0$$

f concave $\Rightarrow$ $x^\emptyset$ is a global maximizer of f !

Exercise IV.4

$$\max \frac{4}{3}x^2 + y + xz \quad \text{s.t} \quad \sqrt{x^2 + y^2 + z^2} \leq 1$$

Constraint set is compact => existence is guaranteed

Strict inequality of the constraint violates optimality => impose equality

Get rid of non-linear transformations (e.g. square root function): square both sides of the constraint

$$\Rightarrow \max \frac{4}{3}x^2 + y + xz \quad \text{s.t.} \quad \underbrace{x^2 + y^2 + z^2 - 1}_{g(x,y,z)} = 0$$

$$\nabla g(x,y,z) = (2x, 2y, 2z)$$

$$= 0 \quad (\text{for singularities})$$

$$\text{if } (x,y,z)' = 0 \quad \text{for which } \|(x,y,z)\| = 0 \neq 1$$

singularity is not a feasible point in the problem, can be disregarded

=> solution will be interior (as there are no border candidates, and existence is guaranteed)

FOC: $\nabla f(x^*, y^*, z^*) = \lambda \nabla g(x^*, y^*, z^*)$ and $g(x^*, y^*, z^*) = 0$

$$[x] \quad \frac{8}{3}x + z = \lambda \cdot 2x$$

$$[y] \quad 1 = \lambda \cdot 2y \quad \Rightarrow \lambda \neq 0, y \neq 0; \; y = 1/2\lambda$$

$$[z] \quad x = \lambda \cdot 2z$$

$$[\lambda] \quad x^2 + y^2 + z^2 - 1 = 0$$

First question: what is definitely not equal to zero here?

Combining [x] and [z]:

$$\frac{8}{3}x + \frac{1}{2\lambda}x = \lambda\, 2x$$

$$\frac{8}{3}x + \frac{1}{2\lambda}x = \lambda 2x$$

$$\Leftrightarrow \left[\frac{8}{3} + \frac{1}{2\lambda} - 2\lambda\right] x = 0 \qquad (*)$$

- $x = 0$! $[z]$ $z = 1/2\lambda \, x = 0$

  $[\lambda]$ $0^2 + y^2 + 0^2 = 1$

  $\rightarrow$ $y = 1$ or $y = -1$

  $\underbrace{\lambda = 1/2y = 1/2}_{\text{not a loc. minimizer } (\lambda > 0)}$ $\underbrace{\lambda = -1/2}_{\text{not a loc. maximizer } (\lambda < 0)}$

  $\rightarrow$ Candidate $p_1^* := (x^*, y^*, z^*) = (0, 1, 0)$

- $x \neq 0$!

$$\rightarrow \frac{8}{3}\lambda + \frac{1}{2} - 2\lambda^2 = 0$$

$$\Leftrightarrow \lambda^2 - 4/3\,\lambda - 1/4 = 0$$

$$\Leftrightarrow \lambda \in \left\{\frac{2}{3} \pm \sqrt{\frac{4}{9} + \frac{1}{4}}\right\}$$

$$= \left\{\frac{2}{3} \pm \sqrt{\frac{16+9}{36}}\right\}$$

$$= \left\{\frac{2}{3} \pm \sqrt{\frac{25}{36}}\right\} = \left\{\frac{4}{6} \pm \frac{5}{6}\right\}$$

$$= \left\{-\frac{1}{6}, \frac{3}{2}\right\}$$

$$= \left\{ -\frac{1}{6}, \frac{3}{2} \right\}$$

not a loc. max. not a loc. min.

$\to$ investigate only $\lambda = 3/2$!

$\underline{\lambda = 3/2:}$ $[y]$ $y = 1/2\lambda = 1/2 \cdot 3/2 = 1/3$

$[z]$ $z = \frac{x}{2\lambda} = \frac{x}{3}$

$[\lambda]$
$\to$ $x^2 + (1/3)^2 + \left(\frac{x}{3}\right)^2 = 1$

$$\Leftrightarrow \underbrace{\left(1 + \frac{1}{9}\right)}_{=10/9} x^2 = 1 - \frac{1}{9} = 8/9$$

$$\Rightarrow x^2 = 8/10 = 4/5 \; ; \quad x = \pm \frac{2}{\sqrt{5}}$$

$$z = \frac{x}{3} = \pm \frac{2}{3\sqrt{5}}$$

$\to$ candidates

$$p_2^* := \left(\frac{2}{\sqrt{5}}, 1/3, \frac{2}{3\sqrt{5}} \right), \quad p_3^* := \left(-\frac{2}{\sqrt{5}}, 1/3, -\frac{2}{3\sqrt{5}} \right)$$

Which one is the global maximizer?

$$f(p_1^*) = \frac{4}{3} 0^2 + 1 + 0 \cdot 0 = 1$$

$$f(p_2^*) = \frac{4}{3}\left(\frac{2}{\sqrt{5}}\right)^2 + 1/3 + \frac{2}{\sqrt{5}} \cdot \frac{2}{3\sqrt{5}}$$

$$f(p_2^*) = \frac{1}{3}\left(\sqrt{15}\right) \quad 3 \quad \sqrt{5} \quad 3\sqrt{5}$$

$$= \frac{4}{3} \cdot \frac{4}{5} + 1/3 + \frac{4}{15}$$

$$= \frac{16}{15} + \frac{5}{15} + \frac{4}{15} = 5/3 > 1$$

$$= f(p_3^*)$$

$\Rightarrow$ p_2^* and p_3^* are the global maximizers, they yield the value $f(p) = 5/3$.