Exercises Ch. 3 Part 2

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Ex III.5

a. $\nabla f(x) = \left(\frac{1}{2}\left(2x_2 + 2x_1\right), \frac{1}{2}\left(3x_2^2 + 2x_1\right)\right)$

$$= \left(x_1 + x_2 ,\ \tfrac{3}{2} x_2^2 + x_1\right)$$

$$H_f(x) = \begin{pmatrix} 1 & 1 \\ 1 & 3x_2 \end{pmatrix}$$

For $v \in \mathbb{R}^2$:

$$v' H_f(x) v = (v_1 \ v_2) \begin{pmatrix} v_1 + v_2 \\ v_1 + 3x_2 v_2 \end{pmatrix}$$

$$= v_1^2 + v_1 v_2 + v_2 v_1 + 3x_2 v_2^2$$

$$= (v_1 + v_2)^2 + 3x_2 v_2^2 - v_2^2$$

$$= (v_1 + v_2)^2 + (3x_2 - 1) v_2^2$$

for $x_2 < 1/3$: for $\underbrace{(v_1, v_2)}_{v^{(1)}} = (1, 0)$:

$$v^{(1)'} H_f(x) v^{(1)} = v_1^2 = 1^2 = 1 > 0$$

for $v^{(2)} = (v_1, v_2) = (0, 1)$:

$$v^{(2)'} H_f(x) v^{(2)} = (3x_2 - 1) v_2^2$$
$$= 3x_2 - 1 \quad < \quad 0$$

for $x_2 \geq 1/3$ $(x_2 > 1/3)$

$$v' H_f(x) v = (v_1 + v_2)^2 + (3x_2 - 1) v_2^2$$

$$\geq 0 \qquad > 0 \quad \text{for } v \neq 0$$

$\Rightarrow f$ is positive (semi-) definite on

$\{x \in \mathbb{R}^2 : x_2 > 1/3\}$ $(\{x \in \mathbb{R}^2 : x_2 \geq 1/3\})$

b.)

$$\nabla u(c,l) = \left(\alpha \left(\frac{l}{c}\right)^{1-\alpha}, (1-\alpha)\left(\frac{c}{l}\right)^{\alpha}\right)$$

$$\frac{\partial^2}{\partial c^2} u(c,l) = \alpha(1-\alpha)\, c^{\alpha-2}\, l^{1-\alpha} \cdot (-1)$$

$$= -\alpha(1-\alpha)\frac{l}{c}\frac{1}{l^{\alpha}}\frac{1}{c^{1-\alpha}}$$

$$\frac{\partial^2}{\partial l^2} u(c,l) = (1-\alpha)\alpha\, c^{\alpha}\, l^{\alpha-1} \cdot (-1)$$

$$= -\alpha(1-\alpha)\frac{c}{l}\frac{1}{l^{\alpha}}\frac{1}{c^{1-\alpha}}$$

$$\frac{\partial^2}{\partial c \partial l} u(c,l) = \alpha(1-\alpha)\, c^{\alpha-1}\, l^{-\alpha}$$

$$= \alpha(1-\alpha)\frac{1}{c^{1-\alpha}}\frac{1}{l^{\alpha}}$$

$$c^{1-\alpha}\,\ell^{\alpha}$$

$$\Rightarrow \quad H_u(c,\ell) = \alpha(1-\alpha)\,\frac{1}{c^{1-\alpha}}\,\frac{1}{\ell^{\alpha}}\begin{pmatrix} -c/\ell & 1 \\ 1 & -\ell/c \end{pmatrix}$$

plugging $\alpha = 1/2$ and $(c_0, \ell_0) = (1,1)$:

$$\nabla u(1,1) = (1/2,\ 1/2)$$

$$H_u(1,1) = 1/4 \cdot \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$T_{1,(1,1)}(c,\ell) = u(1,1) + \nabla u(1,1)\cdot\begin{pmatrix} c-1 \\ \ell-1 \end{pmatrix}$$

$$= 1 + (1/2,\ 1/2)\begin{pmatrix} c-1 \\ \ell-1 \end{pmatrix}$$

$$= 1 + 1/2\,c - 1/2 + 1/2\,\ell - 1/2$$

$$= 1/2\,(c+\ell)$$

$$T_{2,(1,1)}(c,\ell) = T_{1,(1,1)}(c,\ell) + \frac{1}{2}\begin{pmatrix} c-1 \\ \ell-1 \end{pmatrix}' H_n(1,1)\begin{pmatrix} c-1 \\ \ell-1 \end{pmatrix}$$

$$= \frac{1}{2}(c+\ell) + \begin{pmatrix} c-1 \\ \ell-1 \end{pmatrix}' \begin{pmatrix} -(c-1)+\ell-1 \\ (c-1)-(\ell-1) \end{pmatrix} \cdot \frac{1}{2}$$

$$= \frac{1}{2}(c+\ell) + \frac{1}{8}\left[(c-1)(\ell-1) - (c-1)^2 + (c-1)(\ell-1) - (\ell-1)^2\right]$$

$$= \frac{1}{2}(c+\ell) + \frac{1}{8}(-1)\left(c-1-(\ell-1)\right)^2$$

$$= \frac{1}{2}(c+\ell) - \frac{1}{8}(c-\ell)^2$$

$$(c,\ell) = \left(\frac{3}{2}, \frac{1}{2}\right)$$

$$T_{1,(1,1)}\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{1}{2}\right) = \frac{1}{2} \cdot 2 = 1$$

$$T_{2,(1,1)}\left(\frac{3}{2}, \frac{1}{2}\right) = 1 - \frac{1}{8}\left(\frac{3}{2} - \frac{1}{2}\right)^2$$

$$= 1 - 1/8 = 7/8 = 0.875$$

$$u\left(3/2, 1/2\right) = \sqrt{3/2} \cdot \sqrt{1/2} = \frac{\sqrt{3}}{2} \approx 0.866$$

$\rightarrow$ error of T_2 is much smaller!

$$(c, l) = (5, 4/5):$$

$$T_{1,(1,1)}(5, 4/5) = \frac{1}{2}\left(5 + 4/5\right) = \frac{1}{2} \cdot \frac{29}{5} = 2.9$$

$$T_{2,(1,1)}(5, 4/5) = 2.9 - 1/8 \left(5 - 4/5\right)^2$$

$$= 2.9 - 1/8 \left(\frac{21}{5}\right)^2 \approx 0.7$$

$$u(5, 4/5) = \sqrt{5} \cdot \sqrt{4/5} = \sqrt{4} = 2$$

Ex. III.6

$$Y = \int_{[0,1]\times[0,1]} A \cdot k^{1-\alpha} l^{\alpha} \, d(k, l)$$

$[0,1] \times [0,1]$

$$= A\left(\int_0^1 k^{1-\alpha}\,dk\right)\left(\int_0^1 \ell^{\alpha}\,d\ell\right)$$

$$= A\left[\frac{1}{2-\alpha}k^{2-\alpha}\right]_0^1\left[\frac{1}{1+\alpha}\ell^{1+\alpha}\right]_0^1$$

$$= A\cdot\frac{1}{2-\alpha}\cdot\frac{1}{1+\alpha}$$

$$= \frac{A}{2+\alpha-\alpha^2}$$

$$\frac{\partial}{\partial\alpha}\left(2+\alpha-\alpha^2\right) = 1-2\alpha \begin{cases} >0 & \text{if } \alpha<\frac{1}{2} \\ <0 & \text{if } \alpha>\frac{1}{2} \end{cases}$$

$\rightarrow$ y increases (decreases) in α if $\alpha \geq 1/2$ $(\alpha \leq 1/2)$