E600 Mathematics

Collection of Exercise Problems

Class of 2022

Preface. This document collects the exercise problems for the class of 2022 for E600 Mathematics. Exercises are either to be solved in self-study at home and discussed in class on the next day, or immediately discussed in "in-class exercise sessions" without students being required to work on the exercises at home in advance. It will be announced in due time which problems you should work on before the sessions that discuss solutions. Some problems are taken from the pool of exercises on the course's website, `https://e600.uni-mannheim.de`, indicated by the label "online". For these exercises, solutions are available on the website. Of course, you should try to solve all self-study problems first without looking at the solution, but for these exercises, you can check if your approach works or what you're missing, which can help you with the remaining problems. If you wish to practice your understanding of the course's contents beyond what we do in class, the website offers additional exercises with solutions for every chapter of the course.

For your self-study work on the exercises, your first goal should always be to understand what the problem is and to develop a rough idea of how to approach it. Doing so will already help a lot as this will make our discussions of the solutions much more accessible. Thus, if you can solve the exercises, all the better, but do not feel frustrated if you can't! It is up to you how much time and effort you spend on the self-study exercises (you do not have to hand in solutions), but keep in mind that a great deal of mathematical knowledge comes only through application and practice. The fewest students understand all concepts immediately during lecture that introduces them, and wrapping your head around formal concepts usually requires to apply them at least a few times. So, the more you work on the self-study problems, the easier you will find it to follow the sessions discussing the solutions, the following lectures, and ultimately also the contents of your MSc coursework.

CHAPTER 0

Exercise 0.1: Notation and Logic

a.) Writing Formal Statements (online)

Write down the following verbal statements in formal notation.
Example. All real numbers are also rational numbers.
Answer: $\forall x \in \mathbb{R} : (x \in \mathbb{Q})$

1. The set A contains the number 5.
2. The set B contains the number 5 but not the number 4.
3. No natural number is strictly negative (that is, strictly smaller than zero).
4. If x is positive and y is negative, then (this implies that) the product $x \cdot y$ is negative.
5. At any multiple of π, the sin-function is equal to zero.
6. For any natural number and any integer, if the integer is positive, then their product is positive.

b.) Negation of Statements

Negate the following statements. Is the negation true (1.-3.)?

1. $\exists n \in \mathbb{N} : n < 0$.
2. $\forall x \in \mathbb{R} : (x - 1 > 0 \Rightarrow x > 0)$.
3. $\forall x \in \mathbb{R} : x \in \mathbb{N}$.
4. $P \vee Q$, where P and Q are arbitrary statements.

Exercise 0.2: Set Theory - Cardinality and Power Set

Remember that the cardinalty of a set A, $|A|$, denotes the number of elements in a set.

1. What is the value of $|\emptyset|$?
2. What is the value of $|\{\emptyset\}|$?
3. If $A = \{1, \pi\}$, what is the value of $|\mathcal{P}(A)|$?
4. Express the value of $|\mathcal{P}(A)|$ as a function of $|A|$.

Exercise 0.3: Functions

a.) A Derivative (online)

Take the derivative of $f : \mathbb{R} \mapsto \mathbb{R}, x \mapsto f(x) = \cos(x)/x^2$.

b.) A Limit (online)

Compute $\lim_{x \to 0} \frac{x}{e^x - 1}$. (Hint: L'Hôpital's rule)

Exercise 0.4: Linear Combination and Induction Proof

The induction proof is a very important analytical tool, and should be your go-to approach whenever you want to prove statements that start with "$\forall n \in \mathbb{N} : \ldots$" or "$\forall n \in \mathbb{N}\backslash\{0\} : \ldots$". The solution to this exercise provides a simple example of the typical proof structure.

Consider an arbitrary set X and a subset $A \subseteq X$, and suppose that A is *closed under linear combination*, that is, for any $x, y \in A$ and any $\lambda, \mu \in \mathbb{R}$, it holds that $z = \lambda \cdot x + \mu \cdot y \in A$. With these premises, show that for all $n \in \mathbb{N}$, any linear combination of n elements of A is also contained in A, i.e. show that for any $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in A$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, $z = \sum_{i=1}^{n} \lambda_i x_i$ is also contained in A: $z \in A$.

Exercise 0.5: Notation and Logic

a.) Validity of Arguments

Consider an argument with structure "Premise 1 and Premise 2 imply Conclusion". Is the argument valid for the given combinations of premises and conclusion?
Hint 1: Validity is given only if the premises *necessarily* imply the conclusion, it does not suffice if the premises do not contradict the conclusion.
Hint 2: Recall that you can use circles to illustrate mathematical arguments.

Nr.	Premise 1	Premise 2	Conclusion
1	All dogs do not meow	Snoopy is a dog	Snoopy does not meow
2	All cats dislike rain	Snoopy dislikes rain	Snoopy is a cat
3	A free person has nothing to lose	A prisoner is not a free person	A prisoner has something to lose
4	If it rains, we don't play outside	We play outside	It's not raining

b.) Quantifiers and Implication (online)

Assess whether the following statements are necessary, sufficient, equivalent, or neither of the previous, for $S := (\forall x \in A : (x - \pi \in \mathbb{Z}))$. You may assume that A is not the empty set, so that it contains at least one element.

1. $\exists x \in A : (x - \pi \in \mathbb{Z})$
2. $\forall x \in A : (x - \pi \in \mathbb{N})$
3. $\exists! x \in A : (x - \pi \in \mathbb{Z})$
4. $\nexists x \in A : (x - \pi \notin \mathbb{Z})$
5. $\nexists x \in A : (x - \pi \notin \{1, 2, 3\})$
6. $A = \{1, 2, 3\}$
7. $A = \{1 + \pi, -1 + \pi\}$
8. $\forall x \in A : (x - 4 \geq 2)$

Exercise 0.6: Set Theory

a.) Set Operations (online)

Compute union, intersection and both set differences for $A = \{1, 3, 5, 7, 9\}$ and $B = \{-1, 0, 1, 2, 3, 4, 5\}$.

b.) Statements related to Sets

Let $A = \{2, 4, 6, 8, 10\}$ and $B = \{1, 3, 5, 7, 9\}$. Which of the following statements are true?

1. $2 \in A$
2. $3 \ni B$
3. $4 \notin B$
4. $A \in \mathbb{N}$
5. $A = \{2n : n \in \mathbb{N} \setminus \{0\}\}$
6. $A \cup B = \mathbb{N}$
7. $A \cup B \subset \mathbb{N}$
8. $A = \{2, 4, 6, 8, 10, 2, 4, 6, 8, 10\}$
9. $A = \{2, 4, 6, 8, 10, \{2, 4, 6, 8, 10\}\}$
10. $B = \{n \in \mathbb{N} : ((\exists m \in \mathbb{N} : n = 2m + 1) \vee n < 10)\}$
11. $B = \{n \in \mathbb{N} : ((\exists m \in \mathbb{N} : n = 2m + 1) \wedge n < 10)\}$
12. $A = [2, 10) \cap \mathbb{N}$

For the first and last statement, if they are false, can you modify them to make them true?

Exercise 0.7: Functions

a.) Codomain and Range

Give an example for a function f for which the codomain is not equal to the range of f.

b.) Image of a Set under a Function

Let $f : X \to Y$ be a function, and let $A \subset X$. If we say that y is an element of $f[A]$, i.e. $y \in f[A]$ what exactly do we know about y?
A. $f(y) \in A$.
B. $f^{-1}(y) \in A$.
C. $y \in X$.

D. For some $x \in A$, it holds that $f(x) = y$.
E. $y \in A$.

c.) Preimage of a Set under a Function

Let $f : X \to Y$ be a function, and let $B \subset Y$. If we say that x is an element of $f^{-1}[B]$, i.e. $x \in f^{-1}[B]$, what exactly do we know about f and x?
A. $f(x) \in B$
B. $\exists y \in B : x = f^{-1}(y)$
C. $x \in B$
D. $f(x) = B$
E. f is invertible.
F. $f^{-1}(B) = x$

d.) Derivative using the Appropriate Rule

Calculate $f'(x)$ for $f : \mathbb{R} \mapsto \mathbb{R}, x \mapsto \sin((2x+4)^2)$.

CHAPTER 1

Exercise I.1: Key Concepts of Vector Spaces

a.) The Scalar Product (online)

Compute the scalar product of v and w when...

1. $v = (2,3)'$ and $w = (-2,9)'$
2. $v = (3,2,\ln(8),-6)'$ and $w = (\ln(2),1,-1,1/4)'$ (*Hint:* $a\ln(b) = \ln(b^a)$)
3. $v = (4,19)'$ and $w = (2,3,5)'$

b.) More Scalar Products

Compute $a \cdot b$ for

1. $a = (2,5,1)'$, $b = (1,1,3)' \in \mathbb{R}^3$
2. $a = (2,0,-3,4)$, $b = (9,-8,7,-6) \in \mathbb{R}^4$

What do you tell your colleague who claims to have found $v \in \mathbb{R}^n$ so that $v \cdot v = -1$?

Exercise I.2: Norm and Metric in Vector Spaces

a.) The Norms we use are actually Norms

Recall the most commonly used norms on $\mathbb{R}^2$:

- 1-norm ("Manhattan"): $||x||_1 = |x_1| + |x_2|$
- 2-norm ("Euclidean"): $||x||_2 = \sqrt{x_1^2 + x_2^2}$
- infinity-norm ("Maximum"): $||x||_\infty = \max\{|x_1|, |x_2|\}$

Show that the Euclidean norm constitutes a norm, that is, verify that it satisfies the three properties that define a norm.
Hint 1: To show a statement of the form "$\forall x \in A : S(x)$" where A is a set and $S(x)$ is a statement making reference to $x \in A$, you begin by choosing an arbitrary $x \in A$ (start your proof by "Let $x \in A$.") and then show that the statement holds for this unspecified x, as this establishes that $S(x)$ would hold for any arbitrary element of A, so $\forall x \in A : S(x)$.

Hint 2: You may use the **Cauchy-Schwarz inequality** for the Euclidean space $(\mathbb{R}^n, ||\cdot||_2)$, which states that for any $x, y \in \mathbb{R}^n$:

$$|x \cdot y| \leq ||x||_2 ||y||_2.$$

Remark: Except for the triangle inequality, the norm property proofs for the other norms are highly analogous. Proofs for the $\mathbb{R}^n$, rather than the $\mathbb{R}^2$, follow the exact same structure and only involve slightly more notation.

b.) Norm Equivalence (online)

A key concept related to norms that the course does not discuss is *norm equivalence*. You may recall that e.g. continuity of a function can be shown only relative to the metric/norm considered, so that a given function may be continuous in one normed space but not in another one that uses a different norm. Similarly, convergence of sequences may depend on the metric/norm chosen. To this end, a very appealing aspect of norm-induced metrics in spaces over $\mathbb{R}^n$ with finite $n \in \mathbb{N}$ is that all of our usual p-norms are *equivalent*, which means that if a function f satisfies continuity or a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges with respect to one norm, they do so with respect to the other as well.

It turns out to be sufficient for norm equivalence that we can reduce the differences in two norms to some constants that matter little in the ε/δ-arguments we use to investigate convergence, continuity and related concepts. Formally, two norms $||\cdot||_a$ and $||\cdot||_b$ on X are said to be equivalent if there exist constants $0 < c \leq C < \infty$ such that

$$\forall x \in X : c||x||_a \leq ||x||_b \leq C||x||_a.$$

Show that all p-norms are equivalent on any $\mathbb{R}^n$, i.e. that $\forall p, q \in \mathbb{N} \cup \{\infty\}$, $||\cdot||_p$ is equivalent to $||\cdot||_q$ on $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

Hint: this is easiest done in two steps: (1) show that any p-norm is equivalent to the maximum norm, and (2) show that if a p- and q-norm are both equivalent to the maximum norm, the p- and q-norm are also equivalent to each other.

Note: This proof requires to find a smart series of inequality statements, and is certainly one of the hardest exercises that will be left for self-study. If you struggle to find an approach, feel free to consult the online solution to get you started.

Exercise I.3: Testing for Set Properties

As set properties refer to metric or normed vector spaces, respectively, you need to choose a distance measure. You can assume (as always when nothing else is explicitly specified) that we deal with Euclidean spaces, so that you can use the metric induced by the Euclidean norm, $\|x-y\|_2 = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$, but feel free to choose any other p-norm (including $p = \infty$) that you may find easier to handle (recall the previous exercise: p-norms are equivalent, it does not matter which one you choose).

Hint: In class, we have come across a number of helpful theorems that can be used to investigate set properties. Especially, it may be useful that *(i)* closedness can be investigated using convergent sequences within a set, and *(ii)* openness can be investigated by closedness of the complement.

a.) A subset of the real line (online)

Let $S_1 := \{x \in \mathbb{R} : x^2 \leq 4\}$. Is this set open, closed, compact and/or convex?

b.) Two dimensions (online)

Let $S_3 := \{x \in \mathbb{R}^2 : x_1 \leq 3\}$. Is this set open, closed, compact and/or convex?

CHAPTER 2

Exercise II.1: Matrix Multiplication

a.) Two Matrices (online)

Determine whether the following matrices exist, and if so, compute them: AB, $B'A'$ and BA for

$$A = \begin{pmatrix} 0 & 2 \\ 3 & -5 \\ -2 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \end{pmatrix}$$

Hint: Be aware of the rules for transposition and matrix operations to take some shortcuts!

b.) Some more Products

Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -4 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -5 & 3 \\ 2 & 4 \end{pmatrix}.$$

Determine whether the following matrices exist, and if so, compute them:

1. AB
2. BA
3. $B'A'$
4. $BA + C$
5. $AB + C$
6. $(AB + C)'$

Hint: Be aware of the rules for transposition and matrix operations to take some shortcuts!

c.) Right-Multiplication of Vectors and Dimensionality

Let A be the matrix as in b.). What $n \in \mathbb{N}$ must we choose so that $x \in \mathbb{R}^n$ can be right-multiplied to A, i.e. as Ax? What about $A'x$?

Exercise II.2: Determinant Rules (online)

For the following matrices, compute the determinant using an appropriate rule.

1. $A = \begin{pmatrix} 3 & 8 \\ 2 & -1 \end{pmatrix}$

2. $B = \begin{pmatrix} 1 & -2 & 4 \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{8} \\ 1 & 2 & 1 \end{pmatrix}$

3. $C = \begin{pmatrix} 0 & 0 & 2 \\ 3 & 1 & -1 \\ 2 & 2 & 4 \end{pmatrix}$

Hint: You can test your understanding of the Laplace method by using an appropriate expansion at 3. (of course, the 3×3 rule is still perfectly fine here as well).

Exercise II.3: Matrix Inversion: Concrete Examples

For the following matrices, perform a test for invertability (using e.g. the determinant) and, if possible, compute the inverse matrix, using either a shortcut theorem or the Gauss-Jordan method:

1. (online) $A = \begin{pmatrix} 3 & 8 \\ 2 & -1 \end{pmatrix}$

2. (online) $B = \begin{pmatrix} -3 & 2 & 4 \\ -6 & 5 & 4 \\ 1 & -1 & 0 \end{pmatrix}$

3. $C = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Exercise II.4: Elementary Matrix Operations

Here, we convince ourselves again that the elementary operations really work in the way we introduced them: Consider the matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

Define the matrix E so as to

1. interchange rows 2 and 3 (call the matrix E_1),
2. multiply rows 1 and 3 with $\lambda = 5 \neq 0$ (call the matrix E_2),
3. subtract two times row 1 from row 2 (call the matrix E_3).

Multiply out EA for E_3 and check that indeed, the respective operation is performed.

Exercise II.5: Laplace Expansion (online)

In the course, we said that we typically deal with matrices of manageable size when computing the determinant, or that the matrix has a convenient structure (triangular, diagonal), where computing the determinant is as simple as multiplying all diagonal elements to obtain the trace. In some applications, however, you may not be that lucky, and revert to the general Laplace rule. Especially if there are zeros in the matrix, this method is still quite easily and quickly applied; this exercise aims to convince you of this fact.

Compute $\det(A)$ when

$$A = \begin{pmatrix} 1 & 3 & 0 & 0 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ 0 & 12\cdot\pi & 3 & -5 & 1 \\ 0 & 2 & 0 & -1 & 1 \\ 1 & 1 & 0 & 2 & 3 \end{pmatrix}$$

Hint: Recall that you can expand by either a row or a column. In practice, it is easiest to choose the row or column with the most zeros, both for the initial expansion of A as well as any further sub-matrix or sub-matrices you have to consider. By the time you reach 3-by-3 matrices, you know a rule that allows to compute the determinant, so that you don't have to expand further.

Exercise II.6: Linear Independence Tests

The rank concept is sometimes perceived to be a bit awkward. In large parts, this comes from the application specificity of many approaches to determine the rank: they may work well in some cases but less well in others - if you came across the concept in your previous studies, this issue will likely seem familiar. A fairly easy way out is the matrix-based independence test we saw in class, which provides a uniformly applicable method to determine the rank. The following exercise practices this very useful test.

Consider the following sets of vectors:

$$S_1 = \left\{ \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 13 \\ 37 \\ 16 \end{pmatrix} \right\}, \quad S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \\ 0 \end{pmatrix} \right\}, \quad S_3 = \left\{ \begin{pmatrix} -2 \\ 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 3 \\ 0 \end{pmatrix} \right\}.$$

are these sets linearly independent?
Hint: Recall that to perform the test, you need to bring the matrix of stacked column vectors to (generalized) triangular form and investigate the rank.

Exercise II.7: Eigenvalues, Definiteness and Invertability

Let $A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$. Determine the eigenvalues of A. What can you say about A's definiteness based on the eigenvalues? Is A invertible? How could you have checked invertability more directly?

CHAPTER 3

Exercise III.1 Important Objects of Differential Calculus

Consider $x_0 \in X \subseteq \mathbb{R}^n$, $f : X \mapsto \mathbb{R}$. What is the type (e.g., real number, matrix, function, operator, etc.) of the following objects:

$$\frac{\partial f}{\partial x_1}(x_0), \qquad \frac{df}{dx}, \qquad \frac{\partial}{\partial x_1}, \qquad \nabla f, \qquad \frac{df(x_0)}{dx}$$

What, if any, is the difference between $\frac{df}{dx}(x_0)$ and $\nabla f(x_0)$?

Exercise III.2: Invertability

a.) Monotonic Functions and Injectivity (online)

Explain verbally or formally why strictly monotonic function is injective.

Hint: It may be helpful to look up the formal definitions of strict monotonicity and injectivitiy. From there, it should not be more than one sentence/line of formula.

b.) Monotonic Functions and Injectivity: Application (online)

Consider the function $f : \mathbb{R} \mapsto \mathbb{R}, x \mapsto \sin(x) - \frac{3}{2}x$. Is this function injective?
Hint: Monotonicity can be investigated from the first derivative.

c.) Some Examples

Determine which of the following functions are invertible, and if they not, which criterion (injectivity or surjectivity) fails. In case of non-invertability, can you restrict the domain and/or codomain to achieve invertability?[1]

[1] You should understand the word "restrict" as cutting the set to the left/right, but do not manipulate it in the middle; e.g. if the initial set is $\mathbb{N}$, then $R = \mathbb{N} \cap [3, 100]$ is an allowed restriction, but $R = \{3, 9, 27, 87\}$ is not.

1.	$f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto \cos(x)$	2.	$f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto x^2$
3.	$f : \mathbb{N} \mapsto \mathbb{N},\ n \mapsto n^4$	4.	$f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto (-1) \cdot \mathbb{1}[x < 1](x-1)^2 + \mathbb{1}[x \geq 1]\log(x)$

Hint: for 4., investigate the two parts of the domain, $x < 1$ and $x \geq 1$, separately, and then think about whether putting the two parts together changes your conclusion. It may also be helpful to draw the function.

Exercise III.3: Convexity and Norm (online)

Is the following set convex? Justify your answer!

$$S := B_\varepsilon(x_0) = \{x \in X : \|x - x_0\| < \varepsilon\}$$

for some $\varepsilon > 0$ (and an arbitrary norm $\|\cdot\|$). What about a closed ball?

Bonus Question: Is the norm function convex, i.e. the function

$$f : \mathbb{R}^n \mapsto \mathbb{R}, x \mapsto \|x\|$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$, $n \in \mathbb{N}$?

Exercise III.4: Differentiation of Matrix Functions (online)

Consider a matrix $A \in \mathbb{R}^{n \times n}$, $n \in \mathbb{N}$.

(i) Show that $\frac{d}{dx}(Ax) = A$.

(ii) What is the derivative of $f : \mathbb{R}^n \mapsto \mathbb{R}, x \mapsto x'Ax$?
Hint: Use *(i)* and the multivariate product rule.

Exercise III.5: Multivariate Differentiation

a.) Support Restriction?

Use the Hessian Criterion to investigate whether the function

$$f(x_1, x_2) = \frac{1}{2}(x_2^3 + 2x_1x_2 + x_1^2)$$

is (strictly) convex or concave on $\mathbb{R}^3$, and otherwise try to find the support restrictions on which one of the properties holds.
Hint: The function is infinitely many times continuously partially differentiable, which can save you a few computational steps.

b.) Multivariate Taylor and Cobb-Douglas

Note: The following exercise practices the multivariate version of Taylor's Theorem. If you are less familiar with the theorem in general and/or you find the multivariate version difficult to access, there is an online exercise on the univariate version of the theorem that may help you get started.

For what follows, consider a household with Cobb-Douglas utility over consumption c and leisure l, i.e. $u(c,l) = c^\alpha l^{1-\alpha}$ with $\alpha \in (0,1)$. You can use that at points $(c,l) \neq (0,0)$, this function is infinitely many times differentiable.

1.: Approximation of Order 1. Compute the Taylor approximation of order 1 to $u(c,l)$ at $(c_0,l_0) = (1,1)$. For $\alpha = 1/2$, compare the approximated values for $(c,l) = (3/2,1/2)$ and $(c,l) = (5,4/5)$ to the true value of u.

2.: Approximation of Order 2. Compute the Taylor approximation of order 2 to $u(c,l)$ at $(c_0,l_0) = (1,1)$. For $\alpha = 1/2$, compare the approximated values for $(c,l) = (3/2,1/2)$ and $(c,l) = (5,4/5)$ to the true value of u.

Write down the first order Taylor expansion. You may use $\lambda \in (0,1)$ as an unknown variable.

Exercise III.6: Multivariate Integration

Consider an economy populated by a mass $[0,1]$ of firms that use capital k and labor l to produce output $y = f(k,l) = Ak^\alpha l^{1-\alpha}$, i.e. they use the same Cobb-Douglas production technology. Further, suppose that economy-wide output satisfies

$$Y = \int_{[0,1]\times[0,1]} f(k,l)d(k,l).$$

Amongst others, this relationship can be obtained from assuming that labor l and capital k are independently and uniformly distributed on $[0,1]$. However, it is not too important what this means here, it just ensures that the equality above holds.

Determine Y as a function of A and α. What do you conclude for the role of α, the relative importance of capital in the production process in terms of its relationship to Y?

Exercise III.7: Intermediate Value Theorem (online)

Here, we consider a new theorem that the lecture had not introduced:

> **Intermediate Value Theorem.** Let $f : X \mapsto \mathbb{R}$ for some set $X \subseteq \mathbb{R}$, and assume that f is continuous. Then, for any $a,b \in X$ with $a < b$ and $f(a) \leq f(b)$ (and $f(a) \geq f(b)$), for any $y \in [f(a),f(b)]$ (for any $y \in [f(b),f(a)]$), there exists $c \in [a,b]$ with $f(c) = y$.

Verbally, this theorem relates to the intuition of being able to draw continuous functions without lifting the pen: if the continuous function attains two different values within the codomain, it will also reach every value in between along the way. The exercise to follow extends this intuition by establishing that for two continuous functions, if one lies above the other at one point but below at another point, then the functions must intersect in between the points.

a.) Intersecting Continuous Functions

Use the intermediate value theorem to show that if two continuous functions f, g with domain $X \subseteq \mathbb{R}$ and codomain $\mathbb{R}$ are such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$ for some $a, b \in X$, then there exists a value $x \in X$ in between a and b (i.e., $x \in [a, b]$ when $a \leq b$ and $x \in [b, a]$ else) such that $f(x) = g(x)$.

b.) Surjectivity

Consider the function $f : \mathbb{R} \mapsto \mathbb{R}, x \mapsto \sin(x) - \frac{3}{2}x$. Is this function surjective? Is it invertible?
Hint: Use the result of a.). We know about injectivity from a previous exercise.

Exercise III.8: Mean Value Theorem

Prove the Mean Value Theorem, i.e. show that for $f \in D^1(X)$, $X \subseteq \mathbb{R}$, for any $a, b \in X$ so that $a < b$, there exists $x_0 \in (a, b)$ so that

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

What does this imply for the existence of "critical values" of f on X, i.e. points $x \in X$ where $f'(x) = 0$? Illustrate this idea graphically.

CHAPTER 4

Exercise IV.1: Optimization Basics

a.) Concepts (online)

(i) Verbally explain the difference and relationship between a maximum and a maximizer.

(ii) Relate the arg max to the maximum value in an optimization problem using a mathematical statement (refer to a function $f : X \mapsto \mathbb{R}$).

(iii) Let $x_1^*, x_2^* \in X$ such that $x_1^* \in \arg\max_{x \in X} f(x)$ and $x_2^* \in \arg\max_{x \in C} f|_C(x)$ where $C \subseteq X$ is a constraint set. Can you have $f(x_2^*) > f(x_1^*)$? Explain why (not).
Hint: Less formally, *(iii)* asks whether you can have a strictly larger value in a constrained optimization problem than in an unconstrained problem with the same objective.

(iv) Can be $\arg\max_{x \in X} f(x)$ empty? Can it have more than one element if there is a strict global maximizer?

b.) Weierstrass: Concrete Examples

Does the Weierstrass Extreme Value Theorem apply to the functions

1. $f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto x^3$
2. $f : (0,\pi) \mapsto \mathbb{R},\ x \mapsto \cos(x)$
3. $f : \{x \in \mathbb{R}^2_+ : (1,2) \cdot x \leq 5\} \mapsto \mathbb{R},\ x \mapsto x_1 + x_2$
4. $f : [-1,1] \mapsto \mathbb{R},\ x \mapsto \mathbb{1}[x > 0]$
5. $f : [0,\pi] \mapsto \mathbb{R},\ x \mapsto (\cos(x)+2)^{\sin(x)}$
6. $f : \bar{B}_1(\mathbf{0}) \mapsto \mathbb{R},\ x \mapsto x'x$ where $\mathbf{0} \in \mathbb{R}^5$

Is there an example in the functions 1.-6. that demonstrates that the Weierstrass Extreme Value Theorem only formulates only a sufficient, but not an equivalent condition for existence of the global extreme values?
Hint: If you get stuck on 3., ask yourself whether the domain resembles a set that you know from economics.

Exercise IV.2: Economic Formulation and Application (online)

Disclaimer: This problem is pretty long. It covers examples for many of economics' most popular optimization problems, including utility maximization, cost minimization, computation of indirect utility functions in a model parameter, welfare maximization and an exchange economy problem, which are all important in their own right, and you should have seen them at least once. Feel free to focus on the ones you find easier to solve, but make sure to understand what every one of the exercises wants.

Suppose we have two individuals, Martin and Anna. Both like to spend their free time performing only two activities: relaxing (r) and going climbing (c). Otherwise, they don't derive utility from any other source. Suppose that an hour of relaxation costs 1 (e.g. for a Netflix account, food and drinks, or whatever you like to consume in your free time), and an hour of climbing costs 10 (equipment, gym subscription, etc.). Suppose that both Martin and Anna are employed at the same job, and can work for a net hourly wage of 15 to generate income; they both have no initial wealth. Their preferences differ: we have

$$u_M(r,c) = r^{\frac{1}{3}} c^{\frac{2}{3}}$$

for Martin and

$$u_A(r,c) = \sqrt{rc}$$

for Anna. This means that Martin puts more weight on climbing whereas both activities are equally weighted for Anna.

(i) Writing Down and Simplifying the Problem

Formulate the problem that Anna faces when maximizing utility within a given day that has 24 hours. Simplify the problem as much as possible.

(ii) Budget Constraint Interpretation

How can you interpret the budget constraint that you obtain after simplification?

(iii) Solving Anna's Problem

Solve Anna's utility maximization problem (finding the optimal distribution of time across activities is enough; the value of utility does not matter).

(iv) A Problem with Savings

Assume now that Anna has some savings and does not need to work on the day we consider in our optimization problem here. Given her utility function, what is the minimum amount of money that she needs to spend to achieve at least the same level of utility as before? (Even though she does not need to work here, she can still not spend more than 24 hours on both activities combined.)
Hint 1: Once you have simplified the problem to have only one choice variable, it may be instructive to investigate whether or not the time budget constraint binds by looking at the first derivative of the objective.
Hint 2: Don't worry if your results don't give nice numbers anymore, you will need a calculator for this exercise. You may round all intermediate results to two digits.

(v) Differences in Results

How do you explain the difference in results of *(iv)* and *(iii)*, especially in terms of the level of expenditures?

(vi) Martin's Wage Problem

How much would Martin need to earn per hour to afford Anna's level of utility if he has no savings? You may not be able to solve for the wage to the cent; thus, assume that the wage is an integer value. You can use without proof that utility is strictly increasing in the wage and check in steps of 1 whether a given wage yields at least the desired level of utility.
Hint: Solve Martin's utility maximization problem in analogy to Anna's with variable wage to derive the maximal utility as a function of the wage in a first step.

(vii) Martin's Bound on Utility

What level of utility can Martin maximally attain as his wage increases? Why is utility not unbounded above? What can you say about Martin's utility and time allocation when he does not have to work from this investigation?

(viii) Welfare Maximization with Tradeable Goods: Problem

Consider the problem where now, c and r are tradeable goods, e.g. cookies and rice (in kg). Suppose that Martin and Anna live on a deserted island, on which there are 10 boxes of cookies and 12 kg of rice. Formulate the welfare-maximizing resource allocation problem when equal weight is given to both individuals and simplify it as much as possible. How do you proceed to find the solution (verbally)?

(ix) Welfare Maximization with Tradeable Goods: Solution

What is the allocation that solves the problem of *(viii)*?
Hint: Beware of border solutions.

(x) Exchange Economy

Finally, consider again the island scenario but now assume that Anna initially owns all the cookies and Martin owns all the rice. Assume that they can freely discuss exchanging the goods, have perfect information about all aspects of the trade and there is no asymmetry in terms of negation power between the two. At what ratio do they trade the goods, and what is the final allocation? How does aggregate welfare compare to the previous exercise?
Hint: You can express both agents' utility to trading a given quantity of goods at a fixed ratio as a function of the ratio and the goods quantity, and then solve for concrete values that sustain an equilibrium by thinking about the condition that ensures that both agents do not want do deviate.

Exercise IV.3: Solution Existence for Univariate Concave Functions

Consider a univariate, real-valued function $f : (\underline{x}, \bar{x}) \mapsto \mathbb{R}$, $\underline{x}, \bar{x} \in \mathbb{R}$ so that $\underline{x} < \bar{x}$. Suppose that
(i) f is once differentiable,
(ii) f is concave, and that
(iii) there exist $a, b, c \in (\underline{x}, \bar{x})$ with $a < b < c$ so that $f(a) < f(c) < f(b)$.
Can you argue that f assumes a global maximum on $(\underline{x}, \bar{x})$?

Hint 1. Recall that concavity is a desirable feature in unconstrained maximization.
Hint 2. Think about combining the Mean Value and Intermediate Value Theorem.

Exercise IV.4: Saving Time in Optimization

Solve

$$\max \frac{4}{3}x^2 + y + xz \quad \text{s.t.} \quad \|(x, y, z)\|_2 \leq 1$$

where $\|\cdot\|_2$ is the Euclidean norm of the $\mathbb{R}^3$.

You will get a multitude solutions to the FOC. For any one of these, using the second order condition, you would have to check two determinants of the Bordered Hessian, one for a 3×3 and one for a 4×4 matrix. What may help to avoid this is that you can make an argument for solution existence, and also the Lagrangian multiplier condition.

Three simplifications and intuitions can be extremely helpful:

- If $\|(x, y, z)\|_2 < 1$, we can increase the objective varying marginally y.
- By norm non-negativity, $\|(x, y, z)\|_2 \leq 1$ is equivalent to $\|(x, y, z)\|_2^2 \leq 1^2$.
- Closed balls of the $\mathbb{R}^n$ are compact.