E600 Mathematics

Chapter 1: Key Concepts in Vector Spaces

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1. Introduction
Outline

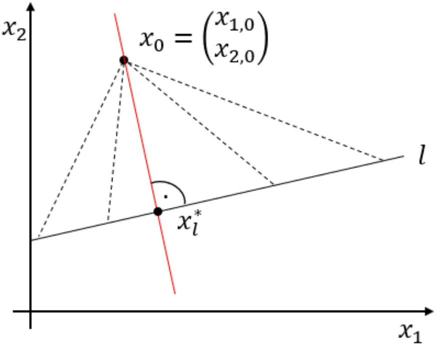
In this chapter, we discuss

- The intuition of the general, formal vector space concept
- Mathematical distance functions and their properties
- Key properties of general sets (open/closed, bounded, convex)
- Limits and continuity beyond univariate real-valued functions

- We are ($\pm$) well-familiar with the mathematics of real numbers, and perhaps also with elements in $\mathbb{R}^2$ (two goods, consumption/leisure, $t = 0$ vs. $t = 1$, etc.)
- The **vector space** concept. . .
  - widely extends the range of objects with which we can do Math (addition, multiplication, etc.) in this familiar fashion
  - transfers graphical intuitions when a geometric picture is not available

$\rightarrow$ Allows discussions of convergence, continuity, differentiability, etc. beyond the real line

- Our focus is $\mathbb{R}^n$ with many dimensions n, but the concept applies also to matrices, functions, and further objects



- Which point x_l^* on the line l minimizes the distance to x_0?

$\rightarrow$ l and the line that connects x_0 and x_l^* are *orthogonal*

- Orthogonality – and the solution approach – is not specific to $\mathbb{R}^2$!
  - Later: formalities of the general orthogonality concept

$\Rightarrow$ Intuition from the $\mathbb{R}^2$ can help to identify solution approaches in more complex settings!

- Vector of length n = ordered $n-$tuple of objects
- Row vs. column vector (convention: "vector" = column vector)
- Real vector of length 2: $x = (x_1, x_2)' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ where $x_1, x_2 \in \mathbb{R}$
- $\mathbb{R}^n := \{(x_1, \ldots, x_n)' : (\forall i \in \{1, \ldots, n\} : x_i \in \mathbb{R})\},\ n \in \mathbb{N}$
- Function vector of length $n \in \mathbb{N}$: $f = (f_1, \ldots, f_n)'$ where f_i, $i \in \{1, \ldots, n\}$ are functions

$\Rightarrow$ Vectors can collect **any kind** of objects and be of **arbitrary** (including zero or infinite) length!

- Vector vs. set: $(2, 2, 3)' \neq (3, 2, 2)'$

- Consider the vectors $x = (0,4)', y = (2,-4)' \in \mathbb{R}^2$ (draw them!)
- Recall from high school: "Directionality and Magnitude"
  - $x = 4 \cdot (0,1)'$, $y = 6 \cdot (1/3, -2/3)'$
  - Fundamental directions of $\mathbb{R}^2$ (*axes*): $e_1 = (1,0)'$ and $e_2 = (0,1)'$
  - Any direction combines them: e.g. $(1/3, -2/3)' = 1/3 \cdot e_1 + (-2/3) \cdot e_2$
- Ingredients:
  - Scalar multiplication: e.g. $4 \cdot (0,1)' = (0,4)' = x$ (scalar?)
  - Vector addition: e.g. $(1/3, 0)' + (0, 2/3)' = (1/3, 2/3)'$
  - Collection of fundamental directions: *basis*
- **General vector space** = collection of set X of vectors + operations for scalar multiplication and vector addition ("basis operations")
  - Basis operations to be defined depending on X, "similar" to $\mathbb{R}^2$
  - All vector spaces generalize most properties and intuitions from $\mathbb{R}/\mathbb{R}^2$!

- Scalar Product: function $\cdot : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}, (x, y) \mapsto x \cdot y = \sum_{i=1}^n x_i y_i$
  - Defined on the *Cartesian product* of $\mathbb{R}^n$ with itself (why?)
  - Alternative notation: $\langle x, y \rangle$ or $x'y$
  - $x \cdot x = \sum_{i=1}^n x_i^2$ ("sum of squares")
  - Also called "dot product", "inner product" or "vector product"
- Orthogonality
  - ...of vectors x, y: $x \cdot y = 0$
  - ...of lines f, g in $\mathbb{R}^n$: $(x_f^1 - x_f^2) \cdot (x_g^1 - x_g^2) = 0$, where x_f^1, x_f^2 are *distinct* points on f and x_g^1, x_g^2 are *distinct* points on g
  - Intuition: see discussion of radian angle in the script/online course

- Consider the following problems:
  1. If $x = (1, 2, 4)'$ and $y = (3, 0, 2)'$, what is $(2x) \cdot y$?
  2. Verbally or formally argue why the following are true for any $x, y \in \mathbb{R}^n$:
     1. $x \cdot y = y \cdot x$
     2. (Binomial Formula): $(x + y) \cdot (x + y) = x \cdot x + 2(x \cdot y) + y \cdot y$
     3. If $x \cdot x = 0$ then $x = \mathbf{0} = (0, 0, \ldots, 0)'$

     (Hint: think about the "sum of squares" property for the last two)

- Linear combination (LC) of ...
  - Two vectors $x, y \in X$: $z = \lambda x + \mu y$, where $\lambda, \mu \in \mathbb{R}$
  - n vectors $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$: $z = \sum_{i=1}^{n} \lambda_i x^{(i)}$ with $\lambda_i \in \mathbb{R}$ $\forall i \in \{1, \ldots, n\}$
- Span = set of linear combinations of $\{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}$, e.g.:

$$Span\left(\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}\right) = \left\{\begin{pmatrix}\lambda+\mu\\\mu\\0\end{pmatrix} : \lambda, \mu \in \mathbb{R}\right\} = \left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \in \mathbb{R}^3 : x_3 = 0\right\}$$

- Basis of a vector space: smallest set to *span* the space
  - Smallest: fewest number of elements
  - Dimension of vector space: number of elements in the basis
  - Basis is not unique: e.g. $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ vs. $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$ for $\mathbb{R}^2$
  - More detail in script/online course

- Linear dependence: $S \subseteq X$ set, $x \in X$ vector. x is linearly dependent of S if it is a LC of elements in S, or equivalently, $x \in Span(S)$
- Linear independence (LI): $x \notin Span(S)$; LI set $B \subseteq X$: no element linearly depends on the remaining set: $\forall b \in B : (b \notin Span(B \backslash \{b\}))$
  - E.g. $\{e_1, e_2\}$: $Span(\{e_1\}) = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\} \not\ni e_2$ and vice versa

Theorem (Testing Linear Independence)

A equivalent condition for the set of vectors $B = \{b_1, b_2, ..., b_k\}$ to be linearly independent is that

$$\sum_{j=1}^{k} \lambda_j b_j = \mathbf{0} \Rightarrow (\forall j \in \{1, ..., k\} : \lambda_j = 0). \tag{1}$$

Applying the LI Test: An Example (Simon & Blume (1994))

The vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n$$

are linearly independent, because if $\lambda_1 e_1 + \ldots + \lambda_n e_n = 0$, i.e.

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \ldots + \lambda_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The last vector equation implies that $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$.

General test: rank of matrix that stacks vectors of B in columns (cf. Chapter 2)

- First concept to generalize from the $\mathbb{R}^2$
- Consider Mannheim, which is, like Manhattan, organized in squares
- If you want to go watch a movie at Cineplex in P4 from the Econ building in L7...
- Generally, what should we intuitively expect from a distance?
  1. Non-negative, and zero only if we don't have to move
  2. Symmetric: same distance from A to B and from B to A
  3. Detours increase the distance
- ...that's exactly what Mathematicians think of a distance as well!

- Generally, what should we intuitively expect from a distance?
  1. Non-negative, and zero only if we don't have to move
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  3. Detours increase the distance

Definition (Metric)

$\mathbb{X} = (X, +, \cdot)$ *real vector space. Then, a function* $d : X \times X \mapsto \mathbb{R}$ *defines a* **metric** *on* X *if*

	Condition	*Name*
(i)	$\forall x, y \in X : d(x,y) \geq 0$, *and* $d(x,y) = 0 \Leftrightarrow x = y$	*non-negativity*
(ii)	$\forall x, y \in X : d(x,y) = d(y,x)$	*symmetry*
(iii)	$\forall x, y, z \in X : d(x,y) \leq d(x,z) + d(z,y)$	*triangle inequality*

...but this general definition may allow some "bugs"/unintuitive behavior! (see script/online course)

Definition (Norm and Normed Vector Space)

$\mathbb{X} = (X, +, \cdot)$ *real vector space. Then, a* function $\|\cdot\| : X \mapsto \mathbb{R}$ *defines a norm on* X *if*

	Condition	*Name*
(i)	$\forall x \in X : \|x\| \geq 0$, *and* $\|x\| = 0 \Leftrightarrow x = \mathbf{0}$	*non-negativity*
(ii)	$\forall x, y \in X : \|x + y\| \leq \|x\| + \|y\|$	*triangle inequality*
(iii)	$\forall x \in X, \lambda \in \mathbb{R} : \|\lambda \cdot x\| = \|\lambda\| \cdot \|x\|$	*absolute homogeneity*

Then, we call $(\mathbb{X}, \|\cdot\|)$ *a* **normed vector space**.

- p-Norm on $\mathbb{R}^n$: $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$, Max. norm: $\|x\|_\infty = \max_{i \in \{1,\ldots,n\}} |x_i|$
- Natural norm on $\mathbb{R}$: $\|x\| = |x|$ (equal to any $\|x\|_p$, including $p = \infty$!)
- Norm-induced metric: $d_N(x, y) = \|x - y\|$ (metric property?)

- Why norm-induced metrics $d_N(x, y) = \|x - y\|$?
  - d_N exhibits the following additional, useful properties:
    1. absolute homogeneity: $\forall x, y \in X \; \forall \lambda \in \mathbb{R} \; d_N(\lambda x, \lambda y) = |\lambda| d_N(x, y)$
    2. translation invariance: $\forall x, y, z \in X \; d_N(x + z, y + z) = d_N(x, y)$
  - Length/magnitude as distance from origin: $\|x\| = \|x - \mathbf{0}\| = d_N(x, \mathbf{0})$
  - Norms are *continuous* and thus analytically tractable
    - Continuity of the norm function $\| \cdot \| : X \mapsto \mathbb{R}, x \mapsto \|x\|$:

$$\forall \varepsilon > 0 \exists \delta > 0 : (\|x - y\| < \delta \Rightarrow \big| \|x\| - \|y\| \big| < \varepsilon)$$

    - Inverse triangle inequality: $\big| \|x\| - \|y\| \big| \leq \|x - y\|$ $\rightarrow$ holds with $\delta = \varepsilon$
    - Intuition: points that are close together ($\|x - y\| < \delta$) have similar magnitude ($\big| \|x\| - \|y\| \big| < \varepsilon$)
    - Continuity $\rightarrow$ we can pull in limits: $\lim_{n \to \infty} \|x_n\| = \| \lim_{n \to \infty} x_n \|$

- Economists typically consider **Euclidean Spaces** $(\mathbb{R}^n, d_N^2)$ with

$$d_N^2(x, y) = \|x - y\|_2 = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2} = \sqrt{(x - y) \cdot (x - y)}$$

  - Crucial importance in Econometrics: Least Squares estimators
  - Geometric intuition: direct distance
  - "The distance" usually refers to the Euclidean norm(-induced metric)

- Goal: have a general concept to measure distances in vector spaces
- Roadmap for *axiomatic* definition
  1. Central intuitive properties of a distance $\rightarrow$ metric concept
     - Large, relatively unstructured class of functions
     - Crude concept, does not rule out some undesirable behaviors
  2. Refinement: **norm-induced metric**
     - Additional properties $\rightarrow$ more narrow concept
     - Fairly simple to define (given that we know some norms)
- Key concept: Euclidean space; measuring direct distances between points in the $\mathbb{R}^n$
- In conclusion, we have ...
  - A preferred way of thinking about distance in the $\mathbb{R}^n$: using norm-induced metrics
  - A similar, useful way of thinking about distance in more general spaces

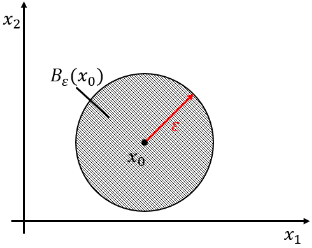
Let's continue our function space example...

- How to define distance of $f(x) = 2\sin(x)$ and $g(x) = cos(x)$?
- Functions: *supremum* norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$
  - Supremum = "generalized maximum", introduced later
  - sup = max whenever max exists (counterex. (0,1); $\sup(0,1) = 1$)
  - $\|f\|_\infty = 2$; $\|g\|_\infty = 1$
  - Distance (draw): $\|f - g\|_\infty = \max_{x \in X} |f(x) - g(x)| = 2$

Finally, a useful trick for norms ("inverse triangle inequality"):

$$\forall x, y \in X : \|x - y\| \geq \big| \|x\| - \|y\| \big|.$$

- Distance functions are fundamentally important for economics
  - Limits and Continuity of general functions are defined using them
  - Related set properties (open, closed, compact) are at the heart of optimization
  - Least squares estimators
  - ...
- Let's begin with the necessary definitions!



- Again: intuition from the $\mathbb{R}^2$
- Ball of radius $\varepsilon > 0$ around x_0: all points with distance to x_0 "smaller" than ε
  - strictly ($d(x, x_0) < \varepsilon$): open ball $B_\varepsilon(x_0)$
  - weakly ($d(x, x_0) \leq \varepsilon$): closed ball $\bar{B}_\varepsilon(x_0)$
  - "closed balls include the boundary, open balls do not"
- Two types of points: interior and boundary points ($int(A)$ vs. ∂A)
- Open set: only interior, no boundary points: $A = int(A)$
- Closed set: also includes all boundary points: $A = int(A) \cup \partial A$
- Bounded set: bounded distance of elements:
  $\exists x \in X \exists r < \infty : A \subseteq B_r(x)$
- Compact set: closed and bounded ("room with walls")

- Concepts are formally a bit tedious, see script for more detail
- Sets may be neither open nor closed (include boundary only partly, e.g. $[a, b)$) or both (no boundary, e.g. $\mathbb{R}$ or $\varnothing$)
- Open/closed interval in $\mathbb{R}$ is open/closed ball:

$$(a, b) = \left(\frac{a+b}{2} - \frac{b-a}{2}, \frac{a+b}{2} + \frac{b-a}{2} \right) = B_{\frac{b-a}{2}} \left(\frac{a+b}{2} \right)$$

- Compact $=$ closed and bounded is actually a theorem (Heine-Borel)
- Compactness is fundamentally important for optimization

- ε-open Ball around x_0:

$$B_\varepsilon(x_0) \overset{\text{generally}}{=} \{x \in X : d(x, x_0) < \varepsilon\}$$
$$\overset{d \text{ norm-induced}}{=} \{x \in X : \|x - x_0\| < \varepsilon\}$$

- Interior point: $x \in int(A) \Leftrightarrow (\exists \varepsilon > 0 : B_\varepsilon(x) \subseteq A)$ (graphically?)
- Boundary point: $x \in \partial A \Leftrightarrow (\forall \varepsilon > 0 : B_\varepsilon(x) \cap A \neq \varnothing \wedge B_\varepsilon(x) \backslash A \neq \varnothing)$
  - $B_\varepsilon(x) \cap A \neq \varnothing$: $B_\varepsilon(x)$ contains some points of A...
  - $B_\varepsilon(x) \backslash A \neq \varnothing$: ...but also some points outside A
- Definitions are a bit awkward, how do we proceed in practice?

Theorem (Properties of Open and Closed Sets)

Consider a metric space $(\mathbb{X}, d)$. Then,

(o.i) $\varnothing$ *and* X *are open in* $\mathbb{X}$.

(o.ii) A set $A \subseteq X$ *is open if and only if its complement* $A^c = X \setminus A$ *is closed.*

(o.iii) The union of an arbitrary (possibly infinite) collection of open sets is open.

(o.iv) The intersection of a finite collection of open sets is open.

(c.i) $\varnothing$ *and* X *are closed in* $\mathbb{X}$.

(c.ii) A set $A \subseteq X$ *is closed if and only if its complement* $A^c = X \setminus A$ *is open.*

(c.iii) The union of a finite collection of closed sets is closed.

(c.iv) The intersection of an arbitrary (possibly infinite) collection of closed sets is closed.

Take-away: check complements and/or decompose into $\cup/\cap$ of simple sets!

Theorem (Closedness and Sequences)

Suppose that $\mathbb{X} = (X, +, \cdot)$ *is a real vector space, and let* $B \subseteq X$. *Then,* B *is closed if and only if, for any convergent sequence* $\{x_n\}_{n\in\mathbb{N}}$ *over* B, *i.e.* $\forall n \in \mathbb{N} : x_n \in B$, *it holds that* $\lim_{n\to\infty} x_n \in B$.

Theorem (Weak Inequalities and the Limit: Functions)

Suppose that $\mathbb{X} = (X, +, \cdot)$ *is a real vector space,* $f : X \mapsto \mathbb{R}$ *and* $g : X \mapsto \mathbb{R}$ *so that* $\forall x \in X$: $f(x) \leq g(x)$ *(in function notation:* $f \leq g$*). Let* $x_0 \in X$, *and suppose that* $\exists f_0, g_0 \in \mathbb{R}$ *so that* $\lim_{x\to x_0} f(x) = f_0$, $\lim_{x\to x_0} g(x) = g_0$. *Then, it holds that* $f_0 \leq g_0$.

Theorem (Weak Inequalities and the Limit: Sequences)

Suppose that $\mathbb{X} = (X, +, \cdot)$ *is a real vector space. Let* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *be convergent sequences over* X, *i.e.* $\forall n \in \mathbb{N} : x_n, y_n \in B$, *with limits* $x \in X$ *and* $y \in X$, *respectively. If* $\forall n \in \mathbb{N}$, *it holds that* $x_n \leq y_n$, *then, we also have* $x \leq y$.

Theorem (Checking Boundedness)

$(\mathbb{X}, d)$ metric space ($\mathbb{X} = (X, +, \cdot)$) where d is norm-induced, i.e. for $x, y \in X$, $d(x, y) = \|x - y\|$. Let $A \subseteq X$. Then, A is bounded if the norm is bounded on A, i.e. $\exists b < \infty : (\forall x \in A : \|x\| < b)$.

Theorem (Budget Set Compactness in the $\mathbb{R}^2$)

Consider the Euclidean space $\mathbb{R}^2$, and the set $B(y|p_1, p_2) := \{x = (x_1, x_2)' \in \mathbb{R}^2_+ : p_1 x_1 + p_2 x_2 \leq y\}$, the budget set with income $y \in \mathbb{R}$ given prices $p_1, p_2 \geq 0$. Then, the budget set is closed, and if $p_1, p_2 > 0$, the budget set is also bounded and thus compact.

- Convergence of a sequence:
  - Recall $\mathbb{R}$: real sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent with limit $x \in \mathbb{R}$ if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : (\forall n \in \mathbb{N}, n \geq N : |x_n - x| < \varepsilon)$$

  - Recall: $|\cdot|$ is the natural norm of the $\mathbb{R}$, so that equivalently

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : (\forall n \in \mathbb{N}, n \geq N : \|x_n - x\| < \varepsilon)$$

  - General **normed** VS $(\mathbb{X}, \|\cdot\|_X)$: $\{x_n\}_{n\in\mathbb{N}}$ where $\forall n \in \mathbb{N} : x_n \in X$ ("sequence over X") is convergent with limit $x \in X$ if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : (\forall n \in \mathbb{N}, n \geq N : \|x_n - x\|_X < \varepsilon)$$

- For sequences over $\mathbb{R}^n$ (cf. online exercises):

$$\lim_{n\to\infty} (x_1, \ldots, x_n) = (\lim_{n\to\infty} x_1, \ldots, \lim_{n\to\infty} x_n)$$

- Convergence of a function:
  - Recall: for a univariate, real-valued function, i.e. $f : X \mapsto Y$ with $X, Y \subseteq \mathbb{R}$, $f_a \in Y$ is the limit of f at $a \in X$ if

$$\forall \varepsilon > 0 \exists \delta > 0 : (\forall x \in X : (|x - a| \in (0, \delta) \Rightarrow |f(x) - f_a| < \varepsilon))$$

  - General function $f : X \mapsto Y$ where $X \subseteq (\mathbb{X}, \| \cdot \|_X)$, $Y \subseteq (\mathbb{Y}, \| \cdot \|_Y)$:

$$\forall \varepsilon > 0 \exists \delta > 0 : (\forall x \in X : (\|x - a\|_X \in (0, \delta) \Rightarrow \|f(x) - f_a\|_Y < \varepsilon))$$

  - Can equivalently write $x \in B_\delta(a) \backslash \{a\}$ for $\|x - a\|_X \in (0, \delta)$
- More general definitions for any metric space (not "just" norm-induced) exist, less relevant to us

- Continuity idea just like before: $f(a) = \lim_{x \to a} f(x)$

$\Rightarrow$ Continuity of f at x_0:

$$\forall \varepsilon > 0 \exists \delta > 0 : (\forall x \in B_\delta(x_0) : \|f(x) - f(x_0)\|_Y < \varepsilon)$$

- Sequence characterization and disproving approach generalizes
  - Limit can be "pulled in": $\lim_{x \to x_0} f(x) = f(\lim_{x \to x_0} x) = f(x_0)$

  $\Rightarrow$ For continuous f, for any sequence $\{x_n\}_{n \in \mathbb{N}}$ with limit x_0, it holds that $\lim_{n \to \infty} f(x_n) = f(\lim_{n \to \infty} x_n) = f(x_0)$
  - Disprove continuity: find $x_n \overset{n \to \infty}{\to} x_0$ with $f(x_n) \overset{n \to \infty}{\not\to} f(x_0)$ (non-existent or different limit)

- You may know convexity of functions (discussed later); here: convexity of *sets*
- Economists are not always fortunate enough to deal with spaces (e.g. budget set is not a space) $\rightarrow$ how to preserve *most* of the structure?
- In a space: *any* linear combination of elements contained
- Convex set: restrict attention to *convex* combinations

Definition (Convex Combination, Convex Set)

$\mathbb{X} = (X, +, \cdot)$ real vector space. A convex combination x^c of the vectors $x_1, \dots, x_n \in X$ is a linear combination $x^c = \sum_{i=1}^n \lambda_i x_i$, for which $\forall i \in \{1, \dots, n\} : \lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$.
A set $A \subseteq X$ is convex if it contains all convex combinations of any two of its elements, i.e. $\forall a_1, a_2 \in A \forall \lambda \in [0, 1] : \lambda a_1 + (1 - \lambda) a_2 \in A$.

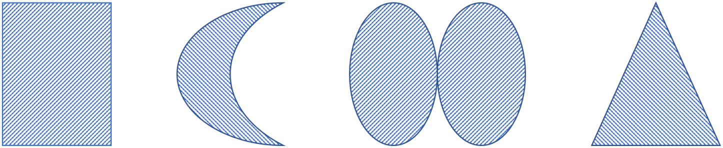
- Iteration: A contains *any* convex combination
- 2 vectors: $\{\lambda x + (1-\lambda)y : \lambda \in [0,1]\}$ = line connecting x and y
  - Intuition: $\lambda x + (1-\lambda)y = y + \lambda(x-y)$

  $\Rightarrow$ The larger λ, the more we move from y to x

$\Rightarrow$ Graphical test in $\mathbb{R}^2$ and $\mathbb{R}^3$: connecting lines fully contained in set?

5. Convexity of Sets

Which Sets are Convex?



- Convex sets in economics: e.g. budget sets (why?)

Proposition (Convexity-preserving Operations)

$\mathbb{X} = (X, +, \cdot)$ *real vector space. Then,*

(i) $\varnothing$ *and* X *are convex.*

(ii) *if* $A \subseteq X$ *is convex, then so is* $\alpha A := \{\alpha \cdot a : a \in A\}$ *for any* $\alpha \in \mathbb{R}$.

(iii) *if* $A, B \subseteq X$ *are convex, then so is* $A + B := \{a + b : a \in A, b \in B\}$.

(iv) *if* $\{A_i\}_{i \in I}$ *is a (possibly infinite) collection of convex sets, then* $\bigcap_{i \in I} A_i$ *is convex.*

(Proof: see script)

Proposition may be helpful for proofs of convexity (decomposition to simpler sets)!

6. Recap Chapter 1

- Defining *basis operations* in similarity to $\mathbb{R}$ and $\mathbb{R}^2$ allows to transfer to more general classes of objects...
  - structured representations of elements (length and magnitude; basis)
  - a broad range of concepts (distance, continuity, etc.)
- Distance functions
  - Main approach to measuring distance: norm-induced metric
  - Economics prefer "direct distances" $\rightarrow$ Euclidean norm & space
- We have learned about helpful set properties
  - open/closed, compact (distance-based – depend on the norm considered)
  - convex (linear combination-based – independent of the norm)
- You may consider the material understood if you understand well slides 5, 16 and 19