Exercise 0

Mittwoch, 24. August 2022 11:06

Ex. 0.1

a)

1. $5 \in A$
2. $5 \in B \wedge 4 \notin B$
   $\uparrow$ "and" (vs. $\vee$: "or")
3. $\nexists n \in \mathbb{N} : (n < 0)$
4. $((x > 0) \wedge (y \leq 0)) \Rightarrow x \cdot y \leq 0$
5. $\forall z \in \mathbb{Z} : (\sin(z\pi) = 0)$
   (alt.: $\forall z\pi \in \mathbb{R} \, (z \in \mathbb{Z} \Rightarrow \sin(z\pi) = 0))$)
6. $\forall n \in \mathbb{N} : (\forall z \in \mathbb{Z} : (z \geq 0 \Rightarrow nz \geq 0))$

b) 1. $\neg (\exists n \in \mathbb{N} : n < 0)$

$\Leftrightarrow \forall n \in \mathbb{N} : \neg (n < 0)$

$\Leftrightarrow \forall n \in \mathbb{N} : n \geq 0$ ✓

2. $\neg(\forall x \in \mathbb{R} : (x-1>0 \Rightarrow x>0))$

$\Leftrightarrow \exists x \in \mathbb{R} : (x-1>0 \quad \wedge \quad x \leq 0)$

$\Leftrightarrow \exists x \in \mathbb{R} : (x>1 \quad \wedge \quad x \leq 0)$

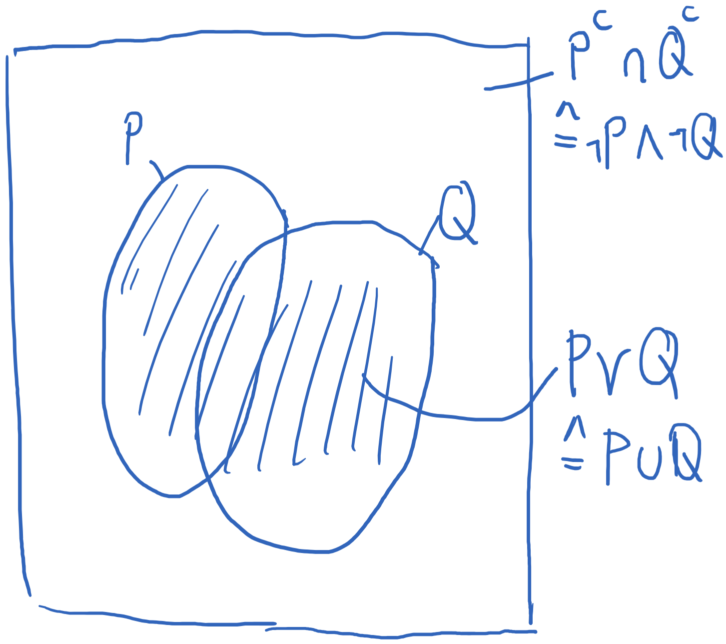
not true!

3. $\neg(\forall x \in \mathbb{R} : x \in \mathbb{N})$

$\Leftrightarrow \exists x \in \mathbb{R} : x \notin \mathbb{N}$ ✓ (e.g. $x = \pi$)

4. $\neg(P \vee Q)$

$\Leftrightarrow \neg P \wedge \neg Q$



<u>Ex 0.2</u>

$\mathcal{P}(A) = \{B : B \subseteq A\}$

by convention: $\emptyset \subseteq A$ for any set A

$|\emptyset| = 0$! $\qquad \emptyset = \{\ \}$

1. $|\emptyset| = 0$! $\qquad \emptyset = \{\ \}$

A: 0 $\qquad$ C: more than 0

B: 1 $\qquad$ D: π

2. $|\{\emptyset\}| = 1$

(e.g. $|\{\{1,2,3\}\}| = 1$ as well!)

3. $A = \{1, \pi\}$

$\mathcal{P}(A) = \dots \left\{ \emptyset, \{1,\pi\}, \{1\}, \{\pi\} \right\}$

$|\mathcal{P}(A)| = 4$

A: 1 $\qquad$ B: 2 $\qquad$ $\boxed{C: 4}$ $\qquad$ D: 16

4. $|\mathcal{P}(A)| = f(|A|)$

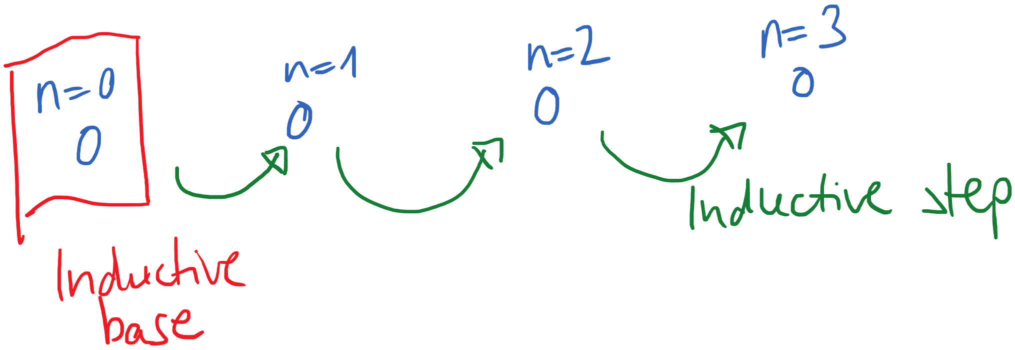
$|\mathcal{P}(\emptyset)| = 1 = 2^0, \quad |\emptyset| = 0$

$|\mathcal{P}(\{1,\pi\})| = 4 = 2^2, \quad |\{1,\pi\}| = 2$

$|\mathcal{P}(B)| = 2^{|B|}$

Ex. 0.4

Induction proof : start with "$\forall n \in \mathbb{N} \ldots$"



Claim: If : $\forall x, y \in A \; \forall \lambda, \mu \in \mathbb{R} : (\lambda x + \mu y \in A)$

then: $\forall n \in \mathbb{N} : \forall x_1, \ldots, x_n \in A \; \forall \lambda_1, \ldots, \lambda_n \in \mathbb{R} :$

$$\left(\sum_{i=1}^{n} \lambda_i x_i \in A \right)$$

Proof: by induction

Base : Let $n = 1$. Let $x_1 \in A$, $\lambda_1 \in \mathbb{R}$.

Then, "$\lambda x + \mu y$"

$$\lambda_1 x_1 = \underbrace{\lambda_1}_{\in \mathbb{R}} \underbrace{x_1}_{\in A} + \underbrace{0}_{\in \mathbb{R}} \cdot \underbrace{x_1}_{\in A} \in A \quad \text{by assumption}$$

$\rightarrow$ Base ✓

<u>Step:</u> Assume that the statement to be shown holds for a fixed $n \in \mathbb{N}$.
("Inductive hypothesis" IH)

Then, let $x_1, \ldots, x_n, x_{n+1} \in A$ and $\lambda_1, \ldots, \lambda_n, \lambda_{n+1} \in \mathbb{R}$.

$$\Rightarrow \sum_{i=1}^{n+1} \lambda_i x_i = \underbrace{\left(\sum_{i=1}^{n} \lambda_i x_i \right)}_{\in A \text{ by IH}} + \left(\lambda_{n+1} x_{n+1} \right)$$

$$= \underbrace{1}_{\in \mathbb{R}} \cdot \underbrace{\left(\sum_{i=1}^{n} \lambda_i x_i \right)}_{\in A} + \underbrace{\lambda_{n+1}}_{\in \mathbb{R}} \cdot \underbrace{x_{n+1}}_{\in A}$$

$$\in A \quad \text{by assumption}$$

$\rightarrow$ step ✓