Solutions 4 - Add-on

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Exercise 1

b. Take the first and second derivative, and judge the definiteness of the Hessian to determine convexity/concavity

$$\nabla f(x) = \left(x_1 + x_2,\ \tfrac{3}{2}x_2^2 + x_1 \right)$$

$$H_f(x) = \begin{pmatrix} 1 & 1 \\ 1 & 6x_2 \end{pmatrix}$$

For $v \in \mathbb{R}^2$:

$$v' H_f(x) v = (v_1 \quad v_2) \begin{pmatrix} v_1 + v_2 \\ v_1 + 6x_2 v_2 \end{pmatrix}$$

$$= v_1^2 + 2v_1 v_2 + 6x_2 v_2^2$$

$$= (v_1 + v_2)^2 + (6x_2 - 1) v_2^2$$

generally indefinite, but if $6x_2 - 1 \geq 0$:

$$v' H_f(x) v \geq 0 \quad \forall v \in \mathbb{R}^n$$

$\Rightarrow$ pos. semi-definite on $\{x \in \mathbb{R}^2 : x_2 \geq 1/6\}$

c. Cobb Douglas and Multivariate Taylor

$$u(c,l) = c^{\alpha} \, l^{1-\alpha}$$

$$\nabla u(c,l) = \left(\alpha \left(\frac{c}{l}\right)^{\alpha-1}, (1-\alpha)\left(\frac{c}{l}\right)^{\alpha} \right)$$

$$H_u(c,l) = \begin{pmatrix} -\alpha(1-\alpha)\left(\frac{c}{l}\right)^{\alpha} \frac{l}{c^2} & \alpha(1-\alpha)\frac{1}{c}\left(\frac{c}{l}\right)^{\alpha} \\ \alpha(1-\alpha)\frac{1}{c}\left(\frac{c}{l}\right)^{\alpha} & -\alpha(1-\alpha)\frac{1}{l}\left(\frac{c}{l}\right)^{\alpha} \end{pmatrix}$$

$$= \alpha(1-\alpha)\left(\frac{c}{l}\right)^{\alpha} \cdot \frac{1}{c} \begin{pmatrix} -l/c & 1 \\ 1 & -c/l \end{pmatrix}$$

approximations here:

$$(c_0, l_0) = (1,1):$$

$$u(1,1) = 1$$

$$\nabla u(1,1) = (\alpha, 1-\alpha)$$

$$H_u(1,1) = \alpha(1-\alpha) \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$T_{1,(1,1)}(c,l) = u(1,1) + \nabla u(1,1) \begin{pmatrix} c-1 \\ l-1 \end{pmatrix}$$

$$= 1 + \alpha(c-1) + (1-\alpha)(\ell-1)$$

$$T_{2,(1,1)}(c,\ell) = T_{1,(1,1)}(c,\ell) + \frac{1}{2}\begin{pmatrix} c-1 \\ \ell-1 \end{pmatrix}' H_u(1,1)\begin{pmatrix} c-1 \\ \ell-1 \end{pmatrix}$$

$$= 1 + \alpha(c-1) + (1-\alpha)(\ell-1) + \frac{\alpha(1-\alpha)}{2} *$$

$$\begin{pmatrix} c-1 \\ \ell-1 \end{pmatrix}' \begin{pmatrix} \ell-1 & -(c-1) \\ c-1 & -(\ell-1) \end{pmatrix}$$

$$= 1 + \alpha(c-1) + (1-\alpha)(\ell-1)$$

$$+ \frac{\alpha(1-\alpha)}{2}\Big((c-1)(\ell-c) + (\ell-1)(c-\ell)\Big)$$

$$= 1 + \alpha(c-1) + (1-\alpha)(\ell-1) - \frac{\alpha(1-\alpha)}{2}(c-\ell)^2$$

At $\alpha = 1/2$:

$$T_{1,(1,1)}(c,\ell) = 1 + \frac{1}{2}(c+\ell-2)$$

$$T_{2,(1,1)}(c,\ell) = 1 + \frac{1}{2}(c+\ell-2) - \frac{1}{8}(c-\ell)^2$$

Evaluating at locations:

$$T_{1,(1,1)}\left(\tfrac{3}{2},\tfrac{1}{2}\right) = 1 + \tfrac{1}{2}\,|2-2| = 1$$

$$T_{2,(1,1)}\left(3/2,\,1/2\right) = 1 - \left(\tfrac{1}{8}\cdot 1^2\right) = \tfrac{7}{8}$$

$$u\left(3/2,\,1/2\right) = \sqrt{3/2 \cdot 1/2} = \tfrac{\sqrt{3}}{2}$$

$$\varepsilon_1 \approx -0{,}134; \qquad \varepsilon_2 \approx 0.009 \quad ; T_2 \text{ better!}$$

$$T_{1,(1,1)}\left(5,\,4/5\right) = 1 + \tfrac{1}{2}\left(5 + 4/5 - 2\right) = \tfrac{29}{10}$$

$$T_{2,(1,1)}\left(5,\,4/5\right) = \tfrac{29}{10} - \tfrac{1}{8}\left(5 - 4/5\right)^2 = 0.695$$

$$u\left(5,\,4/5\right) = \sqrt{5 \cdot 4/5} = 2$$

$$\Rightarrow T_1 \text{ better!}$$

"Far" from the approximation point; the higher order approximation need not do better!

Exercise 4c

1. not bounded; Weierstrass does not apply
2. not closed; Weierstrass does not apply
3. looks like budget set = compact; Weierstrass applies
4. not continuous: jumps at x=0; Weierstrass does not apply
5. compact + continuous function; Weierstrass applies
6. compact + continuous function; Weierstrass applies

4. is an example of sufficiency: function still assumes global max and min on the domain.