Solutions 4

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Exercise 1

$$Y = \int_{I = [0,1] \times [0,1]} f(k,l)\, d(k,l)$$

$$= \int_I A\, k^\alpha l^{1-\alpha}\, d(k,l)$$

$$\overset{\text{Mult. sep functions}}{=} A \left(\int_0^1 k^\alpha\, dk \right) \left(\int_0^1 l^{1-\alpha}\, dl \right)$$

$$= A \left[\frac{1}{1+\alpha} k^{1+\alpha} \right]_0^1 \cdot \left[\frac{1}{2-\alpha} l^{2-\alpha} \right]_0^1$$

$$= A \left[\left(\frac{1}{1+\alpha} k^{1+\alpha} \right) \Big|_{k=1} - \left(\frac{1}{1+\alpha} k^{1+\alpha} \right) \Big|_{k=0} \right]$$

$$\cdot \left[\frac{1}{2-\alpha} 1^{2-\alpha} - \frac{1}{2-\alpha} 0^{2-\alpha} \right]$$

$$= A \cdot \left(\frac{1}{1+\alpha} \cdot 1 \right) \cdot \left(\frac{1}{2-\alpha} \cdot 1 \right)$$

$$= \underline{\underline{\frac{A}{(1+\alpha)(2-\alpha)}}}$$

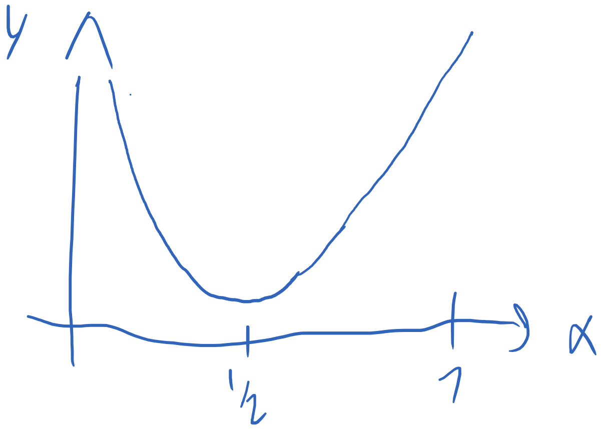
$$\frac{\partial Y}{\partial \alpha} = \frac{\partial}{\partial \alpha} \left\{ A \cdot (2 + \alpha - \alpha^2)^{-1} \right\}$$

$$= \underbrace{A}_{} \cdot \underbrace{(2 + \alpha - \alpha^2)^{-2}}_{} \cdot (-1)(1 - 2\alpha)$$

for $\alpha > \frac{1}{2}$

$$= \underbrace{A}_{>0} \cdot \underbrace{(2 + \ldots)}_{>0} \ldots$$

$$\propto (-1)(1-2\alpha) = 2\alpha - 1 \begin{cases} > 0 & \text{for } \alpha > \frac{1}{2} \\ < 0 & \text{for } \alpha < \frac{1}{2} \end{cases}$$



=> the more equally the inputs are weighted, the lower is the output Y

Why?

as $\alpha \to 1$, $f(k,l) \to Ak$

$\alpha \to 0$, $f(k,l) \to Al$

for factor levels in [0,1], linearly using one factor gives a higher output than multiplicatively combining factors:

$$l > kl \; ; \quad k > kl \qquad \text{for } k, l \in [0,1)$$

Exercise 3

$$u_M(r,c) = r^{1/3} c^{2/3}$$

$$u_A(r,c) = r^{1/2} c^{1/2}$$

(i) Formulating the problem

$$\max_{r,c,h} \underbrace{u_A(r,c)}_{= f(r,c,h)} \quad \text{s.t.} \quad \begin{aligned} 10c + 1r &\leq 15h \\ c + r + h &\leq 24 \\ c, r, h &\geq 0 \end{aligned}$$

Simplifications:

1. not spending all budget is not optimal -> BC binds
2. doing nothing (i.e. spending time on other things than c, r, h) is never optimal -> TC binds
3. any non-negativity constraint binding gives a zero utility => not optimal

$$\hookrightarrow \max_{r,c,h} u_A(r,c) \quad \text{s.t.} \quad \begin{aligned} 10c + 1r &= 15h \\ c + r + h &= 24 \end{aligned}$$

Eliminate choice variables by solving constraints for them:

$$TC \to \quad h = 24 - c - r$$

$$\Rightarrow BC: \quad 10c + 1r = 15(24 - c - r)$$

$$\Leftrightarrow \underbrace{(10+15)}_{25}c + \underbrace{(1+15)}_{16}r = 15 \cdot 24$$

$$25c + 16r = 360 = 24 \cdot \text{wage}$$

RHS = earnings when only working

LHS = opportunity cost of activities

=> potential budget vs opportunity cost budget constraint

Deriving unconstrained problem:

$$26c = 360 - 16r$$

$$26n \qquad 16 \qquad r$$

$26c = 360 - 16r$

$$\Leftrightarrow \quad c = \frac{360}{25} - \frac{16}{25} r$$

$$= \frac{72}{5} - \frac{16}{25} r$$

$$c > 0: \quad \frac{16}{25} r < \frac{72}{5}$$

$$\Leftrightarrow \quad r < \frac{72}{16} \cdot \frac{25}{5} = 5 \cdot \frac{9}{2} = \frac{45}{2} = 22{,}5$$

$$u_A\left(\frac{72}{5} - \frac{16}{25} r, r\right)^2 = \left(\frac{72}{5} - \frac{16}{25} r\right) \cdot r = \frac{72}{5} r - \frac{16}{25} r^2$$

$$\hookrightarrow \max_r \quad \frac{72}{5} r - \frac{16}{25} r^2 \quad \text{s.t.} \quad r \in (0, 22{,}5)$$

(iii) FOC: $\frac{72}{5} - \frac{32}{25} r = 0$

SOC: $\frac{d u_A}{dr}(r) = -\frac{32}{25} < 0$ ✓

(concave ~~convex~~ $\rightarrow$ FOC gives glob. max.)

$$r = \frac{72}{5} \cdot \frac{25}{32} = 5 \cdot \frac{9}{4} = \frac{45}{4} = 11.25 \quad (\in (0, 22.5))$$

$$c = \frac{72}{5} - \frac{16}{25} r = \frac{72}{5} - \frac{\overset{4}{\cancel{16}}}{\cancel{25}} \cdot \frac{\overset{9}{\cancel{45}}}{\cancel{4}} \ldots$$

$$c = \frac{72}{5} - \frac{16}{25} r = \frac{72}{5} - \frac{\cancel{16}}{\cancel{25}_5} \cdot \frac{\cancel{45}}{\cancel{4}_1}$$

$$= \frac{72}{5} - 4 \cdot \frac{9}{5}$$

$$= \frac{36}{5} = 7.2$$

$$(c_A^*, r_A^*) = (7.2, 11.25)$$

=> verify solution by checking that the budget clears with equality

(iv) $$u_A(c_A^*, r_A^*) = \sqrt{\frac{\cancel{36}^9}{\cancel{5}_1} \cdot \frac{\cancel{45}^9}{\cancel{4}_1}} = \sqrt{9^2} = 9$$

$$\min_{c,r} \; 10c + 1r \quad \text{s.t.} \quad \underline{\underline{\sqrt{rc} \geq 9}}, \quad c + r \leq 24, \quad c, r \geq 0$$

Non-negativity constraints: if they bind, they necessarily violate the utility target $\sqrt{rc} \geq 9$

=> can not bind at the optimum

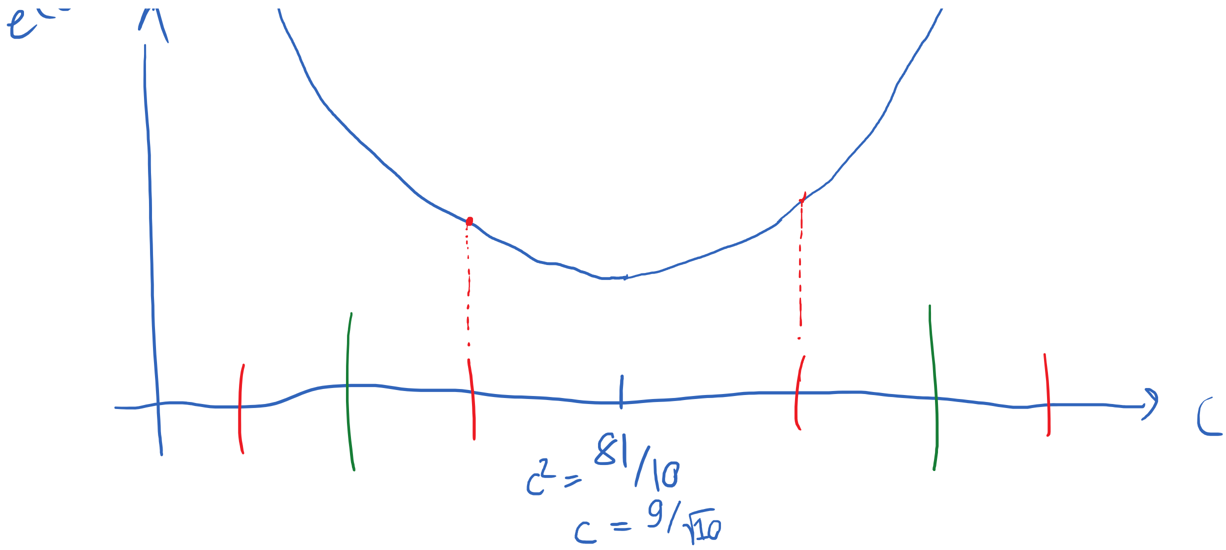
Strictly exceeding the utility target does not give you minimal expenditures => this constraint binds

$$\sqrt{rc} = 9 \iff rc = 81 \iff r = 81/c$$

$$\rightarrow \min_{r} \; 10c + 81/c \quad \text{s.t.} \quad c + 81/c \leq 24$$

Idea: exploit the shape of the objective

$$\frac{d}{dc}\left[10c + \frac{81}{c}\right] = 10 - \frac{81}{c^2} \begin{cases} > 0 & \text{for } c^2 > \frac{81}{10} \\ < 0 & \text{for } c^2 < \frac{81}{10} \end{cases}$$



For a constraint set to the left (right) of 81/10, the largest (smallest) value is the solution

If 81/10 is contained in the constraint set, it is the solution.

81/10 feasible? X

$$\left(c + \frac{81}{c}\right)\Big|_{c = 9/\sqrt{10}} = \frac{9}{\sqrt{10}} + \frac{81}{9}\sqrt{10}$$

$$= \frac{9}{\sqrt{10}} + 9 \cdot \sqrt{10} \approx 31.31 > 24$$

=> solution is not at the interior, but at the boundary! (existence is guaranteed)

Solve $c + \frac{81}{c} = 24$ for c:

$$c^2 - 24c + 81 = 0$$

$$\rightsquigarrow c \in \{4.06, 19.94\}$$

$(c, r) = (4.06, 19.94) \rightarrow$ solution! r is cheaper!

vs. $(c, r) = (19.94, 4.06) \rightarrow$ X

expenditures: $4.06 \cdot 6 + 19.94 \cdot 1 = 40.6 + 19.94$
$= 60.54$

vs. (iii): $7.2 \cdot 10 + 11.25 \cdot 1 = 83.25$

$\Rightarrow$ spend $\geq 25\%$ less for same level of utility

(solution would be identical if we imposed the time budget of (iii), i.e. 24-h*)

(viii)

$$W(c,r) = u_A(c_A, r_A) + u_M(c_M, r_M)$$
$$= \sqrt{c_A r_A} + c_M^{2/3} r_M^{1/3}$$

$$\max W(c,r) \quad \text{s.t.} \quad 10 \geq c_A + c_M$$
$$12 \geq r_A + r_M$$
$$c_A, c_M, r_A, r_M \geq 0$$

throwing away resources is not welfare-maximizing; resource constraints bind

"Border" solutions from the inequalities: all resources given to $i \in \{A, M\}$

while no resources are given to $j \in \{A, M\}$, $j \neq i$

(strictly splitting up resources not optimal: zero utility for both individuals)

$$\max \sqrt{(10 - c_M)(12 - r_M)} + c_M^{2/3} r_M^{1/3}$$

(ix) Look for interior solutions, if any, and compare to border candidates

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$$\begin{cases} [c_M] & -\frac{1}{2}\sqrt{\frac{12-r_M}{10-c_M}} + \frac{2}{3} c_M^{2/3-1} r_M^{1/3} = 0 \\ [r_M] & -\frac{1}{2}\sqrt{\frac{10-c_M}{12-r_M}} + \frac{1}{3} c_M^{2/3} r_M^{\frac{1}{3}-1} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} [c_M] & \frac{4}{3}\left(\frac{r_M}{c_M}\right)^{1/3} = \sqrt{\frac{12-r_M}{10-c_M}} \\ [r_M] & \frac{3}{2}\left(\frac{r_M}{c_M}\right)^{2/3} = \sqrt{\frac{12-r_M}{10-c_M}} \end{cases}$$

" $a = a$ "

" $b = b$ "

$\rightarrow$ $ab = ab$

$a/b = a/b$

(a) $[c_M] \cdot [r_M]$ gives

$$\frac{4}{3} \cdot \frac{3}{2} \frac{r_M}{c_M} = \frac{12-r_M}{10-c_M} \Leftrightarrow \frac{12-r_M}{r_M} = 2 \frac{10-c_M}{c_M}$$

(b) $[r_M] / [c_M]$ gives

$$\frac{3/2}{4/3}\left(\frac{r_M}{c_M}\right)^{2/3-1/3} = 1$$

$$\Leftrightarrow c_M^{1/3} = 9/8 \; r_M^{1/3}$$

$$c_M = (9/8)^3 \; r_M$$

plugging (b) to (a)

$$\frac{12 - r_M}{r_M} = 2\,\frac{10 - (9/8)^3 r_M}{(9/8)^3 r_M} \quad \Big|\cdot (9/8)^3 r_M$$

$$(9/8)^3 (12 - r_M) = 20 - 2\cdot (9/8)^3 r_M$$

$$\Leftrightarrow \quad (9/8)^3 r_M = 20 - (9/8)^3\, 12$$

$$r_M = \frac{20 - (9/8)^3\, 12}{(9/8)^3} \approx 2.05$$

$$c_M = (9/8)^3 r_M \approx 2.91$$

$$\rightarrow \quad r_A = 12 - 2.05 = 9.95$$

$$c_A = 10 - 2.91 = 7.09$$

$$W(\text{"interior solution"}) = \sqrt{9.95 \cdot 7.09} + 2.91^{2/3}\, 2.05^{1/3}$$

$$\approx 10.99$$

$$W(\text{"boundary"}) = \begin{cases} 10.63 & M\text{-boundary} \\ 10.96 & A\text{-boundary} \end{cases}$$

$\Rightarrow$ interior solution gives global maximum!

(x) In the exchange economy, agents must decide on a reference good and express all prices/values relative to this good

→ let $p_r = 1$; p_c as the price of cookies expressed in units of rice

x = trade volume of cookies (non-unit price good)

Allocation:

$c_A = 10 - x$ $\quad$ $c_M = x$

$r_A = p_c \cdot x$ $\quad$ $r_M = 12 - p_c x$

$$u_A(x) = \sqrt{(10-x) p_c x} = \sqrt{p_c}\sqrt{10x - x^2} \qquad u_M(x) = x^{2/3}(12 - p_c x)^{1/3}$$

Optimal trade volume: maximizes both agents' utility from trade

$$\text{FOC}_A: \quad \frac{du_A}{dx}(x) = \frac{\sqrt{p_c}}{2} \frac{10-2x}{\sqrt{10x - x^2}} = 0 \iff \underline{\underline{x = 5}}$$

$$\text{FOC}_M: \quad \frac{du_M}{dx}(x) = \frac{d}{dx}\left((12 - p_c x) x^2\right)^{1/3}$$

$$= \frac{24x - p_c \cdot 3x^2}{\left((12 - p_c x) x^2\right)^{2/3}} \cdot \frac{1}{3} = 0 \iff 8x - p_c x^2 = 0$$

plugging $x = 5$ into FOC_M:

$$8 \cdot 5 - 25 p_c = 0 \iff p_c = 40/25 = \underline{\underline{1.6}}$$

$\rightarrow$ trade volume nice: $p_c x = 1.6 \cdot 5 = 8$

$c_A = 10 - 5 = 5 = c_M$

$r_A = 8$; $r_M = 12 - 8 = 4$

$(c_A, r_A) = (5, 8)$; $(c_M, r_M) = (5, 4)$

level of welfare lower than in welfare-maximizing exercise (obviously), but the resulting allocation is more equal