In-class 2

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Problem 1

$$\det\begin{pmatrix} 1 & 3 & 0 & 0 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ 0 & 12\pi & 3 & -5 & 1 \\ 0 & 2 & 0 & -1 & 1 \\ 1 & 1 & 0 & 2 & 3 \end{pmatrix}$$

$$= 3\cdot\underbrace{(-1)^{3+3}}_{=1}\cdot\det\begin{pmatrix} 1 & 3 & 0 & 0 \\ -1 & 1 & 2 & 1 \\ 0 & 2 & -1 & 1 \\ 1 & 1 & 2 & 3 \end{pmatrix}$$

$$= 3\cdot\left(1\cdot(-1)^{1+1}\det\begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix} + 3\cdot(-1)^{1+2}\det\begin{pmatrix} -1 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\right)$$

$$= 3\begin{pmatrix} -3+2+4 \\ -(-1+12+2) \end{pmatrix} - 3\cdot\begin{pmatrix} 3+2+0 \\ -(-1+0+(-2)) \end{pmatrix}$$

$$= 3\left(3 - 13 \quad - 3(5+3)\right)$$

$$= 3\,(-10 - 24) = -3 \cdot 34 = \underline{\underline{-102}}$$

<u>Pr. 2</u>

$$\ker A = \{x \in \mathbb{R}^n : Ax = \theta\}$$

$$A(\lambda x + \mu y) = \lambda \underbrace{Ax}_{=\theta} + \mu \underbrace{Ay}_{=\theta} = \lambda\theta + \mu\theta = \theta$$

if $x, y \in \ker A$, then also $\lambda x + \mu y \in \ker A$

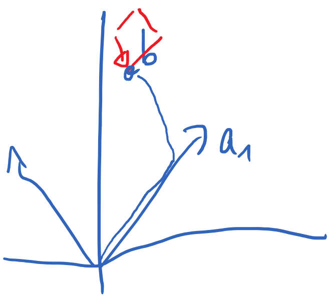
<u>a.</u> Consider the system $Ax = b$ with solution x^*

<u>Claim:</u> $Ax^s = b \iff x^s - x^* = x_0 \in \ker A$

Proof: $Ax^s = b \,(= Ax^*) \iff \theta = Ax^s - Ax^* = A(x^s - x^*)$

$$= Ax_0$$

$\iff x_0 \in \ker A$. $\square$



x_0

1. unique solution: x^*

$\to$ x^S that is a solution is $x^* + \theta$

$\to$ $\ker(A) = \{\theta\}$; $\dim(\ker(A)) = 0$

in case $\dim(\ker(A)) = 1$:

$\ker(A) = \{\lambda v : \lambda \in \mathbb{R}\}$ for some $v \in \mathbb{R}^n$

$\to$ solutions $x^S = x^* + \underline{\underline{\lambda}} \cdot v$

$\Rightarrow$ 1 free variable (λ)

generally, there will be as many free variables in the system as the dimension of the kernel

2.

space of solutions:

$$S(Ax=b) = \{x^* + x_0 : x_0 \in \ker A\}$$

$$= \{x^* + \sum_{j=1}^{d} \lambda_j b_j \text{ for } \lambda_1, \dots, \lambda_d \in \mathbb{R}\}$$

every basis vector b_j of the kernel captures a free dimension/free variable of the problem

b.

when A has m columns, then

$$\dim(\ker(A)) = m - rk(A)$$

$$rk(A) = m - \dim(\ker A)$$

nb. of restricted variables — nb. of all variables — nb. of free variables

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

$$x^* = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ solves } Ax = b$$

General procedure:

1. find a solution ✓
2. find the kernel of A
3. represent the set of solutions using 1. and 2.

$$Ax = 0 \,! \quad \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} x_1 - x_3 \\ 2x_1 + 3x_2 + x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x_1 = x_3$$

$$\rightarrow \quad 2x_3 + 3x_2 + x_3 = 0$$

$$\Leftrightarrow \quad 3x_2 + 3x_3 = 0$$

$$\Leftrightarrow \quad x_2 = -x_3$$

$$\Rightarrow \quad x = x_3 \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} ; \quad \ker A = \operatorname{span}\left(\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} \right)$$

$$= \left\{ \lambda \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

$$\rightarrow \quad S(Ax=b) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

$$\Rightarrow \quad S(Ax=b) = \left\{ \underbrace{\begin{pmatrix} \vdots \\ 1 \\ 1 \end{pmatrix}}_{x^0} + \lambda \cdot \begin{pmatrix} \vdots \\ -1 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$