In-class chapter 1

Mittwoch, 25. August 2021 10:13

Problem 1

$$S_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_2 = 0 \right\} = \underline{\left\{ \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix} : x_1, x_3 \in \mathbb{R} \right\}}$$

subspace: R^3 is a real vector space; S_2 is a subspace if it is closed under linear combination

Let $x, y \in S_2$, $\lambda, \mu \in \mathbb{R}$. Then,

$$\lambda x + \mu y = \begin{pmatrix} \lambda x_1 + \mu y_1 \\ \lambda \cdot 0 + \mu \cdot 0 \\ \lambda x_3 + \mu \cdot y_3 \end{pmatrix} = \begin{pmatrix} \lambda x_1 + \mu y_1 \\ 0 \\ \lambda x_3 + \mu \cdot y_3 \end{pmatrix} \in S_2 .$$

✓

"proper": $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^3$, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \notin S_2$

$$B(S_2) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \{ e_1, e_3 \}$$

$$\operatorname{Span}(B(S_2)) = \{ \lambda e_1 + \mu e_3 : \lambda, \mu \in \mathbb{R} \} = \left\{ \begin{pmatrix} \lambda \\ 0 \\ \mu \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}$$

$$= S_2$$

B(S_2) is a basis of S_2; it contains two elements, therefore the dimension of S_2 is equal to two.

<u>b.)</u>

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$B_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$B_1 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \underbrace{\lambda}_{4} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \underbrace{\mu}_{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \underbrace{\xi}_{5} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Pr. 2

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2}$$

(i) $\forall x \in \mathbb{R}^2 : \ \|x\|_2 \geq 0$ (by $(\cdot) \geq 0$ and $\sqrt{\cdot} \geq 0$)

if $\sqrt{x_1^2 + x_2^2} = 0$, then

$$x_1^2 + x_2^2 = 0 \iff (x_1^2 = 0 \wedge x_2^2 = 0)$$

$$\iff (x_1 = 0 \wedge x_2 = 0)$$

$$\iff x = 0$$

✓

(iii) $\forall x \in \mathbb{R}^2, \ \lambda \in \mathbb{R}$:

$$\|\lambda x\|_2 = \sqrt{(\lambda x_1)^2 + (\lambda x_2)^2} = \sqrt{\lambda^2 (x_1^2 + x_2^2)}$$

$$= \sqrt{\lambda^2} \cdot \sqrt{x_1^2 + x_2^2}$$

$$= |\lambda| \cdot \|x\|_2$$

✓

(ii) $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x \circ x}$

For $x, y \in \mathbb{R}^2$:

$\|x+y\|^2$ ... $= (x+y) \cdot (x+y)$

$$\|x+y\|_2^2 = \left(\sqrt{(x+y)\cdot(x+y)}\right)^2 = (x+y)\cdot(x+y)$$

$$= x\cdot x + 2\,x\cdot y + y\cdot y$$ by bin. formula for scalar product

$$= \|x\|_2^2 + 2\,x\cdot y + \|y\|_2^2$$ by norm definition

$$\leq \|x\|_2^2 + 2|x\cdot y| + \|y\|_2^2$$

C-S inequality

$$\leq \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2$$

$$= \left(\|x\|_2 + \|y\|_2\right)^2$$

so in summary

$$\|x+y\|_2^2 \leq \left(\|x\|_2 + \|y\|_2\right)^2$$

$$\Leftrightarrow \sqrt{\|x+y\|_2^2} \leq \sqrt{\left(\|x\|_2 + \|y\|_2\right)^2}$$

$$\Leftrightarrow \|x+y\|_2 \leq \|x\|_2 + \|y\|_2$$

(ii) Δ-ineq. for $\|\cdot\|_\infty$

For $x, y \in \mathbb{R}^2$,

$$\|x+y\|_\infty = \max\{|x_1+y_1|, |x_2+y_2|\}$$

$$\leq \max\{|x_1|+|y_1|, |x_2|+|y_2|\}$$

$$\leq \max\{|x_1| + \underbrace{\max\{|y_1|,|y_2|\}}_{\|y\|_\infty}, |x_2| + \underbrace{\max\{|y_1|,|y_2|\}}_{\|y\|_\infty}\}$$

$$= \max\{|x_1| + \|y\|_\infty, |x_2| + \|y\|_\infty\}$$

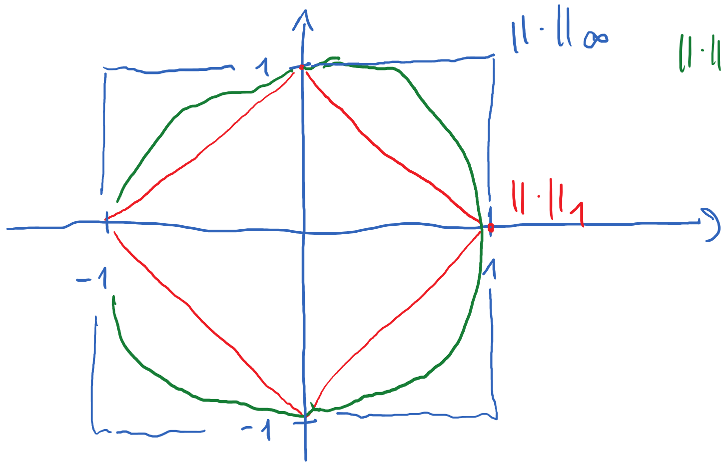
$$= \max\{|x_1| + \|y\|_\infty ,\ |x_2| + \|y\|_\infty\}$$

$$= \max\{|x_1|, |x_2|\} + \|y\|_\infty = \|x\|_\infty + \|y\|_\infty \ \checkmark$$

where in the last step, we used

$$\max\{a+c, b+c\} = \max\{a,b\} + c \quad \forall a,b,c \in \mathbb{R}.$$

(iii)



$\|\cdot\|_2$: $x_1^2 + x_2^2 \le 1$

$$|x_1| \le \sqrt{1 - x_2^2}$$

the larger p in the p-norm, the larger the ball, i.e. the more points included (and conversely, the smaller the measured distance)

$\|\cdot\|_p,\ p \in \mathbb{N}$

b.)

$$\forall x,y \in X: \ \|x-y\| \geq \big|\, \|x\| - \|y\| \,\big|$$

use "false zeros" in the proof!

$$\|x\| = \|x + \underbrace{y - y}_{=0}\| \overset{\Delta\text{-ineq.}}{\le} \|y\| + \|x-y\|$$

$$\Leftrightarrow \|x\| - \|y\| \le \|x-y\|, \qquad \text{and}$$

$$\|y\| = \|y + x - x\| \le \|x\| + \|\underbrace{y-x}_{=(-1)\cdot(x-y)}\| = \|x\| + \underbrace{|-1|}_{=1} \cdot \|x-y\|$$

$$\Leftrightarrow \|y\| - \|x\| \le \|x-y\|$$

$$\big|\, \|x\| - \|y\| \,\big| = \begin{cases} \|x\| - \|y\| & \text{if } \|y\| \le \|x\| \\ \ldots \end{cases} \le \|x-y\|.$$

$$\big| \|x\| - \|y\| \big| = \big\rbrace \|y\| - \|x\| \quad \text{else}$$

c.) for general f: " $\forall \varepsilon > 0 \ \exists \delta > 0 : (\|x-y\| < \delta \Rightarrow | f(x) - f(y) | < \varepsilon)$

here: $f(\cdot) = \|\cdot\|$;

if $\delta > \|x-y\| \overset{\text{inv. } \Delta\text{-ineq.}}{\geq} \big| \|x\| - \|y\| \big|$, then

for $\varepsilon = \delta$, $\varepsilon > \big| \|x\| - \|y\| \big|$

$\Rightarrow \forall \varepsilon > 0 \ \exists \delta = \varepsilon$ s.th. $(\|x-y\| < \delta \Rightarrow \big| \|x\| - \|y\| \big| < \varepsilon)$

$\square$

delta-statement: x and y are "really close to each other"
epsilon-statement (for the norm): x and y are of similar length/magnitude
=> expect intuitively that if delta-statement holds, then also the epsilon-statement will