Solutions 3 - Add-on

Dienstag, 31. August 2021 17:00

Exercise 1 a.) Eigenvalues, Definiteness and Invertability

Eigenvalues: $\lambda \in \mathbb{R}$ where $(A - \lambda I)x = 0$ does **not** have a unique solution

$$\Rightarrow \det(A - \lambda I) \overset{!}{=} 0$$

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} 3-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$$

$$= (3-\lambda)(1-\lambda) - 4$$

$$= 3 - 4\lambda + \lambda^2 - 4 = \lambda^2 - 4\lambda - 1$$

$$= 0$$

$$\Leftrightarrow \lambda \in \{2 \pm \sqrt{4+1}\}$$

$$= \{2 \pm \sqrt{5}\}$$

$$\lambda_1 = 2 + \sqrt{5} > 0, \quad \lambda_2 = 2 - \sqrt{5} < 2 - \sqrt{4} = 0$$

$\Rightarrow$ indefinite (cf. theorem relating sign of eigenvalues to definiteness) nvertible : $\lambda_1, \lambda_2 \neq 0 \Rightarrow$ ✓

more directly, we could have checked whether $\det(A) \neq 0$.

Exercise 1: Linear independence tests

Concept is explained again in notes for solutions discussed in class; here you can find the analytical procedure for the two sets that were not shown

S_1 : stacking the vectors as rows into a matrix gives

$$M_1 = \begin{pmatrix} 1 & 2 & 13 \\ -2 & 3 & 37 \\ 4 & 4 & 16 \end{pmatrix}$$

clear out the first column using the 1 in the top left corner; i.e. apply

II = II + 2 * I
III = III - 4 * I

$$\longrightarrow \begin{pmatrix} 1 & 2 & 13 \\ 0 & 7 & 63 \\ 0 & -4 & -36 \end{pmatrix}$$

getting ones in the second column for rows II and III (multiplying with 1/7 and -1/4, respectively) gives

$$\begin{pmatrix} 1 & 2 & 13 \\ 0 & 1 & 9 \\ 0 & 1 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 9 \end{pmatrix}$$

$\to$ rk $M_1 = 2$; not LI set

S_3 : stacking the vectors as rows into a matrix gives

$$M_3 = \begin{pmatrix} -2 & 2 & 3 \\ 2 & -1 & -1 \\ 2 & 1 & 3 \\ 4 & -1 & 0 \end{pmatrix}$$

use row 2 to eliminate non-zero entries in the first column
(or any other row that you like)

$$\to \begin{pmatrix} 0 & 1 & 2 \\ 2 & -1 & -1 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix}$$

use row 1 to eliminate non-zero entries in row 3 and 4

$$\to \begin{pmatrix} 0 & 1 & 2 \\ 2 & -1 & -1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \operatorname{rk} M_3 = 2$; not LI set

Exercise 3: Invertability

b.) some examples

1. not invertible; both surjectivity and injectivity fail; the restriction to any interval of length pi with codomain [-1,1] is invertible, e.g.

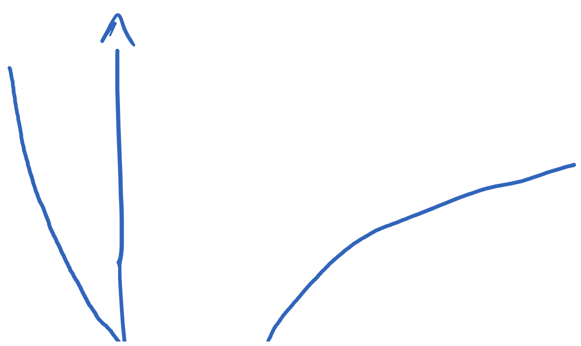
   $f: [0,\pi] \mapsto [-1,1], \; x \mapsto \cos(x)$

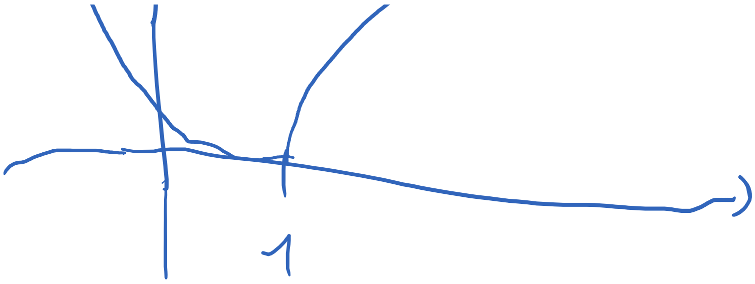
2. both conditions fail (surjectivity: negative real numbers not mapped onto; injectivity: f(-x) = f(x) for all $x \in \mathbb{R}$).
   Restriction:

   $f: \mathbb{R}_+ \mapsto \mathbb{R}_+, \; x \mapsto x^2$

3. surjectivity fails; a lot of $m \in \mathbb{N}$ have no $n \in \mathbb{N}$ for which $m = n^4$
   no restriction applicable

4. not invertible; both conditions fail; graphically





the individual components however are invertable with the appropriate domain and codomain; i.e.

$$f_1: (-\infty, 1) \mapsto \mathbb{R}_+, \quad x \mapsto (x-1)^2$$

$$f_2: [1, \infty) \mapsto \mathbb{R}_+, \quad x \mapsto \log(x)$$

If you were to pre-multiply either component function with (-1), you would get an invertible composite function; e.g.

$$g(x) = \mathbb{1}[x<1](x-1)^2 - \mathbb{1}[x>1]\log(x)$$

or $h(x) = -g(x)$.