Homework 3

Dienstag, 31. August 2021 11:45

Exercise 1

Idea LI test:

1. The rank does not change when applying elementary operations
2. The rank tells me about the number of lin. indep. rows/columns

S_2 : $M_2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 2 & 1 & 3 \\ 4 & -1 & 0 \end{pmatrix}$; investigate rank

$M_2 \xrightarrow[\mathrm{IV} = \mathrm{IV} - 4 \cdot \mathrm{I}]{\mathrm{III} = \mathrm{III} - 2 \cdot \mathrm{I}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 0 & -3 & -3 \\ 0 & -9 & -12 \end{pmatrix}$

$\mathrm{I} = \mathrm{I} + 2 \cdot \mathrm{II}$

$\xrightarrow[\mathrm{IV} = \mathrm{IV} - 9 \cdot \mathrm{II}]{\mathrm{III} = \mathrm{III} - 3 \cdot \mathrm{II}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & -12 \end{pmatrix}$

=> triangular structure without zeros on the diagonal

=> full rank; 3 linearly independent columns

=> S_2 is a linearly independent set

Exercise 2

1. first partial derivative of f at location x_0; real number (level 3)

2. derivative **function** of f (level 2)

3. Operator for the first partial derivative (level 1)

4. Gradient function (level 2)

5. nothing ("wrong" notation)

difference between derivative function and gradient function when both are evaluated at x_0:
- if the function is differentiable, then there is no difference
- gradient of f at x_0 may exist even if the function is not differentiable at x_0, i.e. the derivative at x_0 doesn't exist

Exercise 4

a.) $S = B_\varepsilon(x_0) = \{x \in X : \|x - x_0\| < \varepsilon\} \quad (*)$

"Cookbook recipe" to analytical proofs (such as convexity proofs)

0. Make clear what needs to be shown and what can be assumed

1. check your toolbox: what options are there to show the result of interest
   a. ex. convexity: raw definition? helpful theorems?
   b. here: no theorems -> use raw definition
   c. narrow down what needs to be shown for the tool used; e.g. definition of convexity: inequality for norms

2. Use the tools to approach the result
   => two options: regular triangle or inverse triangle
   => experiment with the options you have

0. Assumption: structure of the set S as

given in (*)

Conclusion / result: S is convex, i.e.

$$\forall x,y \in S \ \forall \lambda \in [0,1]: \ \lambda x + (1-\lambda) y \in S$$

Let $x,y \in S$ and $\lambda \in [0,1]$.

$\Rightarrow \|x - x_0\| < \varepsilon$ and $\|y - x_0\| < \varepsilon$

to show: $\|\lambda x + (1-\lambda) y - x_0\| < \varepsilon$

$\rightarrow$ use triangle inequality

(only result that bounds the norm above)

$$\|\lambda x + (1-\lambda) y - x_0\| = \|\lambda (x - x_0) + (1-\lambda)(y - x_0)\|$$

$$\leq \|\lambda (x - x_0)\| + \|(1-\lambda)(y - x_0)\|$$ by triangle inequality

$$= |\lambda| \, \|x - x_0\| + |1-\lambda| \, \|y - x_0\|$$ by abs. homogeneity

$$\overset{\lambda \in [0,1]}{=} \lambda \|x - x_0\| + (1-\lambda) \|y - x_0\|$$

$$< \lambda \varepsilon + (1-\lambda) \varepsilon$$ by setup/initial assumption

$$= \varepsilon$$

In conclusion

$\|\lambda x + (1-\lambda) y - x_0\| < \varepsilon$, i.e. $\lambda x + (1-\lambda) y \in B_\varepsilon(x_0)$

=> S is convex.

b.) Assumption: $f: \mathbb{R}^n \mapsto \mathbb{R}$, $x \mapsto \|x\|$

Conclusion: f is convex, i.e. $\forall x, y \in X \; \forall \lambda \in [0,1]$:

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y).$$

In analogy to a.): for $x, y \in X$, $\lambda \in [0,1]$:

$$f(\lambda x + (1-\lambda) y) = \| \lambda x + (1-\lambda) y \|$$

$$\overset{\text{a. (with } x_0 = 0)}{\leq} \lambda \cdot \|x\| + (1-\lambda) \|y\|$$

$$= \lambda f(x) + (1-\lambda) f(y).$$

=> f is convex.

What remains to be investigated:

(i) concavity

(ii) strict convexity

(i) Hint: $\iff$ linear, i.e. $\|x+y\| = \|x\| + \|y\|$?

here: RHS > 0 $\iff$ LHS > 0 ?

for $x, y \neq 0$: $\|x\| > 0$, $\|y\| > 0$

$\rightarrow$ RHS $= 0$ $\iff$ $x = y = 0$

LHS $= 0$ $\iff$ $x + y = 0$ $\iff$ $x = -y$

these statements are clearly not equivalent

=> should be possible to build a simple argument for non-equality

Let $x \in X \setminus \{0\}$. Then,

$$\|x - x\| = 0 \neq 2 \cdot \|x\| = \|x\| + \|x\| \cdot |-1|$$
$$= \|x\| + \|-x\|$$

$\Rightarrow$ norms are not linear; $f(\cdot)$ is not concave.

(ii) $\quad f(\lambda x + (1-\lambda)y) \qquad \lambda f(x) + (1-\lambda)f(y)$

$$\to \|\lambda x + (1-\lambda)y\| < \lambda\|x\| + (1-\lambda)\|y\|$$

for $x \neq y$, $\lambda \notin \{0,1\}$ $\quad (\lambda \in (0,1))$

for $x = c \cdot y$, $\quad c \neq 1 \qquad (x \neq y)$

$$\|\lambda x + (1-\lambda)y\| = \|(\lambda c + (1-\lambda))y\|$$
$$= |\lambda c + (1-\lambda)| \cdot \|y\|$$
$$\overset{\text{if } c > 0}{=} \lambda c\|y\| + (1-\lambda)\|y\|$$
$$= \lambda\|cy\| + (1-\lambda)\|y\|$$
$$= \lambda\|x\| + (1-\lambda)\|y\|$$

How did we find this solution?

Starting from what we know about norms: defining properties (non-neg., abs. homogeneity and triangle inequality)

an inequality is not helpful in establishing equality, as needed here

so we are left with the other two, and it turns out that abs. homogeneity can be used to get rid of the "+" on the LHS, and give equality.

Exercise 5 a)

$\exp: \mathbb{R} \to \mathbb{R}_+ , \; x \mapsto \exp(x) = e^x \quad ; \quad x_0 = 1$

$T_{1,1}(x) = f(1) + f'(1)(x-1)$

$T_{1,2}(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$

$\Rightarrow T_{1,1}(x) = e + e(x-1) = ex$

$T_{1,2}(x) = ex + \frac{e}{2}(x-1)^2 = ex + \frac{e}{2}(x^2 - 2x + 1)$

$= \frac{e}{2}x^2 + \frac{e}{2}$

$x_1 = -5:$

$T_{1,1}(x) = -5e \quad , \quad T_{2,1}(x) = \frac{e}{2}((-5)^2+1) = 13e$

$\to \exp(-5) \approx 0$

$|\varepsilon_{11}(-5)| \approx 5e$

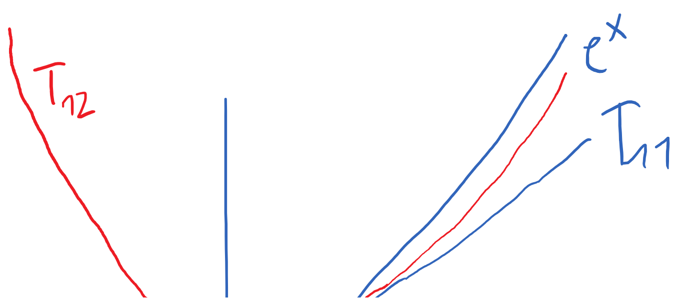
$|\varepsilon_{12}(-5)| \approx 13e > |\varepsilon_{11}(-5)|$

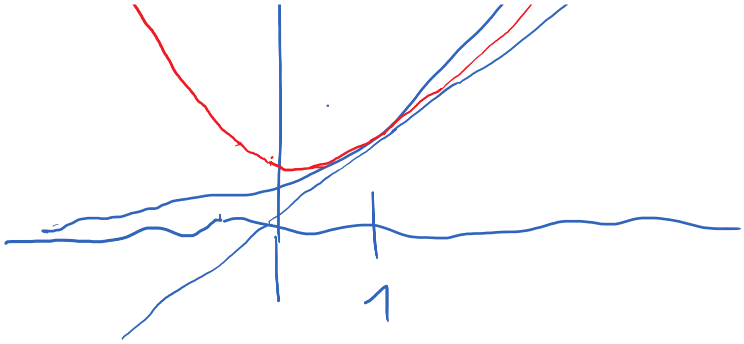
$x_2 = 2$

$T_{1,1}(x_2) = 2e \quad \text{vs.} \quad T_{12}(x) = \frac{e}{2}(2^2+1) = \frac{5}{2}e$

$2.7 \cdot 2.7 = \exp(2) > \frac{5}{2}e = 2.5 \cdot 2.7$

$\to |\varepsilon_{12}(2)| < |\varepsilon_{11}(2)|.$




1