E600 Mathematics

Fall Semester 2021

Self-study Exercises 4

to be solved on Wednesday, September 1

Topics: Integration, Optimization

Exercise for Chapter 3

Exercise 0: Multivariate Differentiation

a.) Hessian Criterion for Convexity (online)

Investigate the following function with respect to (strict) convexity/concavity:

$$f : \mathbb{R}^2 \mapsto \mathbb{R}, x = (x_1, x_2)' \mapsto \exp(x_1) + x_1 x_2 + 5x_1 + 4$$

Hint: Recall that we can use the second derivative to investigate convexity. The function is infinitely many times continuously partially differentiable, which can save you a few computational steps.

b.) Support Restriction?

Use the Hessian Criterion to investigate whether the function

$$f(x_1, x_2) = \frac{1}{2}(x_2^3 + 2x_1 x_2 + x_1^2)$$

is (strictly) convex or concave on $\mathbb{R}^3$, and otherwise try to find the support restrictions on which one of the properties holds.
Hint: The function is infinitely many times continuously partially differentiable, which can save you a few computational steps.

c.) Multivariate Taylor and Cobb-Douglas

For what follows, consider a household with Cobb-Douglas utility over consumption c and leisure l, i.e. $u(c,l) = c^\alpha l^{1-\alpha}$ with $\alpha \in (0,1)$. You can use that at points $(c,l) \neq (0,0)$, this function is infinitely many times differentiable.

1.: Approximation of Order 1. Compute the Taylor approximation of order 1 to $u(c,l)$ at $(c_0,l_0) = (1,1)$. For $\alpha = 1/2$, compare the approximated values for $(c,l) = (3/2,1/2)$ and $(c,l) = (5,4/5)$ to the true value of u.

2.: Approximation of Order 2. Compute the Taylor approximation of order 2 to $u(c,l)$ at $(c_0,l_0) = (1,1)$. For $\alpha = 1/2$, compare the approximated values for $(c,l) = (3/2,1/2)$ and $(c,l) = (5,4/5)$ to the true value of u.

Write down the first order Taylor expansion. You may use $\lambda \in (0,1)$ as an unknown variable.

Exercise 1: Multivariate Integration

Consider an economy populated by a mass $[0,1]$ of firms that use capital k and labor l to produce output $y = f(k,l) = Ak^{\alpha}l^{1-\alpha}$, i.e. they use the same Cobb-Douglas production technology. Further, suppose that economy-wide output satisfies

$$Y = \int_{[0,1]\times[0,1]} f(k,l)d(k,l).$$

Amongst others, this relationship can be obtained from assuming that labor l and capital k are independently and uniformly distributed on $[0,1]$. However, it is not too important what this means here, it just ensures that the equality above holds.

Determine Y as a function of A and α. What do you conclude for the role of α, the relative importance of capital in the production process in terms of its relationship to Y?

Exercise for Chapter 4

Exercise 2: Optimization Basics

a.) Concepts (online)

(i) Verbally explain the difference and relationship between a maximum and a maximizer.

(ii) Relate the arg max to the maximum using a mathematical statement (refer to a function $f : X \mapsto \mathbb{R}$).

(iii) Let $x_1^*, x_2^* \in X$ such that $x_1^* \in \arg\max_{x\in X} f(x)$ and $x_2^* \in \arg\max_{x\in C} f|_C(x)$ where $C \subseteq X$ is a constraint set. Can you have $f(x_2^*) > f(x_1^*)$? Explain why (not).
Hint: Less formally, this question asks whether you can have a strictly larger value in a constrained optimization problem than in an unconstrained problem with the same objective.

(iv) Can be $\arg\max_{x\in X} f(x)$ empty? Can it have more than one argument if there is a strict global maximizer?

b.) Continuity and Weierstrass (online)

Give an example of a discontinuous function on a compact domain that does not have a global maximum.

Hint: Think about the intuition that Chapter 4 has discussed. You may want to define a "split function", i.e. $f(x) = \begin{cases} f_1(x) & \text{if} \dots \\ f_2(x) & \text{else} \end{cases}$, either explicitly or using an indicator term.
Comment: The online solution presents one of many examples. If you find a different one, as the solution also states, that's perfectly fine!

c.) Weierstrass: Concrete Examples

Does the Weierstrass Extreme Value Theorem apply to the functions

1. $f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto x^3$
2. $f : (0, \pi) \mapsto \mathbb{R},\ x \mapsto \cos(x)$
3. $f : \{x \in \mathbb{R}^2_+ : (1,2) \cdot x \leq 5\} \mapsto \mathbb{R},\ x \mapsto x_1 + x_2$
4. $f : [-1,1] \mapsto \mathbb{R},\ x \mapsto \mathbb{1}[x > 0]$
5. $f : [0, \pi] \mapsto \mathbb{R},\ x \mapsto (\cos(x) + 2)^{\sin(x)}$
6. $f : \bar{B}_1(\mathbf{0}) \mapsto \mathbb{R},\ x \mapsto x'x$ where $\mathbf{0} \in \mathbb{R}^5$

Is there an example in the functions 1.-6. that demonstrates that the Weierstrass Extreme Value Theorem only formulates only a sufficient, but not an equivalent condition for existence of the global extreme values?
Hint: If you get stuck on 3., ask yourself whether the domain resembles a set that you know from economics.

Exercise 3: Economic Formulation and Application (online)

Disclaimer: This problem is pretty long. It covers examples for many of economics' most popular optimization problems, including utility maximization, cost minimization, computation of indirect utility functions in a model parameter, welfare maximization and an exchange economy problem, which are all important in their own right, and you should have seen them at least once.

Suppose we have two individuals, Martin and Anna. Both like to spend their free time performing only two activities: relaxing (r) and going climbing (c). Otherwise, they don't derive utility from any other source. Suppose that an hour of relaxation costs 1 (e.g. for a Netflix account, food and drinks, or whatever you like to consume in your free time), and an hour of climbing costs 10 (equipment, gym subscription, etc.). Suppose that both Martin and Anna are employed at the same job, and can work for a net hourly wage of 15 to generate income; they both have no initial wealth. Their preferences differ: we have

$$u_M(r, c) = r^{\frac{1}{3}} c^{\frac{2}{3}}$$

for Martin and

$$u_A(r, c) = \sqrt{rc}$$

for Anna. This means that Martin puts more weight on climbing whereas both activities are equally weighted for Anna.

(i) Writing Down and Simplifying the Problem

Formulate the problem that Anna faces when maximizing utility within a given day that has 24 hours. Simplify the problem as much as possible.

(ii) Budget Constraint Interpretation

How can you interpret the budget constraint that you obtain after simplification?

(iii) Solving Anna's Problem

Solve Anna's utility maximization problem (finding the optimal distribution of time across activities is enough; the value of utility does not matter).

(iv) A Problem with Savings

Assume now that Anna has some savings and does not need to work on the day we consider in our optimization problem here. Given her utility function, what is the minimum amount of money that she needs to spend to achieve at least the same level of utility as before? (Even though she does not need to work here, she can still not spend more than 24 hours on both activities combined.)
Hint 1: Once you have simplified the problem to have only one choice variable, it may be instructive to investigate whether or not the time budget constraint binds by looking at the first derivative of the objective.
Hint 2: Don't worry if your results don't give nice numbers anymore, you will need a calculator for this exercise. You may round all intermediate results to two digits.

(v) Differences in Results

How do you explain the difference in results of *(iv)* and *(iii)*, especially in terms of the level of expenditures?

(vi) Martin's Wage Problem

How much would Martin need to earn per hour to afford Anna's level of utility if he has no savings? You may not be able to solve for the wage to the cent; thus, assume that the wage is an integer value. You can use without proof that utility is strictly increasing in the wage and check in steps of 1 whether a given wage yields at least the desired level of utility.
Hint: Solve Martin's utility maximization problem in analogy to Anna's with variable wage to derive the maximal utility as a function of the wage in a first step.

(vii) Martin's Bound on Utility

What level of utility can Martin maximally attain as his wage increases? Why is utility not unbounded above? What can you say about Martin's utility and time allocation when he does not have to work from this investigation?

(viii) Welfare Maximization with Tradeable Goods: Problem

Consider the problem where now, c and r are tradeable goods, e.g. cookies and rice (in kg). Suppose that Martin and Anna live on a deserted island, on which there are 10 boxes of cookies and 12 kg of rice. Formulate the welfare-maximizing resource allocation problem when equal weight is given to both individuals and simplify it as much as possible. How do you proceed to find the solution (verbally)?

(ix) Welfare Maximization with Tradeable Goods: Solution

What is the allocation that solves the problem of *(viii)*?
Hint: Beware of border solutions.

(x) Exchange Economy

Finally, consider again the island scenario but now assume that Anna initially owns all the cookies and Martin owns all the rice. Assume that they can freely discuss exchanging the goods, have perfect information about all aspects of the trade and there is no asymmetry in terms of negation power between the two. At what ratio do they trade the goods, and what is the final allocation? How does aggregate welfare compare to the previous exercise?
Hint: You can express both agents' utility to trading a given quantity of goods at a fixed ratio as a function of the ratio and the goods quantity, and then solve for concrete values that sustain an equilibrium by thinking about the condition that ensures that both agents do not want do deviate.