E600 Mathematics

Fall Semester 2021

Self-study Exercises 3

to be solved over the weekend, August 28-30

Topics: Linear Independence Test, Multivariate Differentiation

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Exercise for Chapter 2

Exercise 0: Eigenvalues and Definiteness

a.) Eigenvalues, Definiteness and Invertability

Let $A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$. Determine the eigenvalues of A. What can you say about A's definiteness? Is A invertible? How could you have checked invertability more directly?

b.) Definiteness (online)

Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

Is A positive or negative (semi-)definite? What can you say about the invertability of A from this fact?

Exercise 1: Linear Independence Tests

The rank concept is sometimes perceived to be a bit awkward. In large parts, this comes from the application specificity of many approaches to determine the rank: they may work well in some cases but less well in others - if you came across the concept in your previous studies, this issue will likely seem familiar. A fairly easy way out is the matrix-based independence test we saw in class, which provides a uniformly applicable method to determine the rank. The following exercise practices this very useful test.

Consider the following sets of vectors:

$$S_1 = \left\{ \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 13 \\ 37 \\ 16 \end{pmatrix} \right\}, \quad S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \\ 0 \end{pmatrix} \right\}, \quad S_3 = \left\{ \begin{pmatrix} -2 \\ 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 3 \\ 0 \end{pmatrix} \right\}.$$

are these sets linearly independent?

Hint: Recall that to perform the test, you need to bring the matrix of stacked column vectors to (generalized) triangular form and investigate the rank.

Exercises for Chapter 3

Exercise 2: Important Objects of Differential Calculus

Consider $x_0 \in X \subseteq \mathbb{R}^n$, $f : X \mapsto \mathbb{R}$. What is the type (e.g., real number, matrix, function, operator, etc.) of the following objects:

$$\frac{\partial f}{\partial x_1}(x_0), \quad \frac{df}{dx}, \quad \frac{\partial}{\partial x_1}, \quad \nabla f, \quad \frac{df(x_0)}{dx}$$

What, if any, is the difference between $\frac{df}{dx}(x_0)$ and $\nabla f(x_0)$?

Exercise 3: Invertability

a.) Monotonic Functions and Injectivity (online)

Show that any strictly monotonic function is injective.
Hint: It may be helpful to look up the formal definitions of strict monotonicity and injectivitiy. From there, it should not be a far way to showing this fact.

b.) Some Examples

Determine which of the following functions are invertible, and if not, which criterion (injectivity or surjectivity) fails. In case of non-invertability, can you restrict the domain and/or codomain to achieve invertability?[1]

1. $f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto \cos(x)$
2. $f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto x^2$
3. $f : \mathbb{N} \mapsto \mathbb{N},\ n \mapsto n^4$
4. $f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto \mathbb{1}[x < 1](x-1)^2 + \mathbb{1}[x \geq 1]\log(x)$

Hint: for 4., investigate the two parts of the domain, $x < 1$ and $x \geq 1$, separately, and then think about whether putting the two parts together changes your conclusion. It may also be helpful to draw the function.

c.) Monotonic Functions and Injectivity: Application (online)

Consider the function $f : \mathbb{R} \mapsto \mathbb{R}, x \mapsto \sin(x) - \frac{3}{2}x$. Is this function injective?
Comment: We will investigate surjectivity in the in-class exercises for this function.

Exercise 4: Convexity

a.) Set Convexity: revisited (online)

Is the following set convex? Justify your answer!

$$S := B_\varepsilon(x_0) = \{x \in X : \|x - x_0\| < \varepsilon\}$$

for some $\varepsilon > 0$. What about a closed ball?

[1] You should understand the word “restrict” as cutting the set to the left/right, but do not manipulate it in the middle; e.g. if the initial set is $\mathbb{N}$, then $R = \mathbb{N} \cap [3, 100]$ is an allowed restriction, but $R = \{3, 9, 27, 87\}$ is not.

b.) Norms (online)

Investigate the following function with respect to (strict) convexity/concavity:

$$f : \mathbb{R}^n \mapsto \mathbb{R}, x \mapsto \|x\|$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$, $n \in \mathbb{N}$.
Hint 1: We know that norms are continuous, but they need not be differentiable. Hence, the criterion for the second derivative (that you may know from the lectures, if we managed to cover it already) is not useful here, and it is instructive to proceed with the "raw" definition of convexity.
Hint 2: The solution of a.) offers some insight into how we should approach this problem.
Hint 3: A function is only both convex and concave if it is linear, i.e. $f(x+y) = f(x) + f(y)$ for *any* possible arguments x, y. Thus, once you showed that f satisfies one of the properties, you can check the other by investigating whether f is/can potentially be linear.

Exercise 5: Multivariate Differentiation

a.) Taylor Approximation: Univariate case (online; slightly modified)

This exercise is meant as a means to get started with Taylor's theorem for those who did not cover it in their undergraduate or need to fresh up their memory. This less notation-intense context should help you deal with multivariate Taylor approximations as used in the next exercise. Feel free to skip ahead to *(ii)* if you are already well-familiar with Taylor approximations.

1.: First and Second Order Approximation. Compute the first and second order Taylor approximations to the exponential function $\exp : \mathbb{R} \mapsto \mathbb{R}_+, x \mapsto \exp(x)$ around $x_{0,1} = 1$. Evaluate both approximations at $x = -5$ and $x = 2$, and compare the approximation quality. Is one always better than the other? Are the approximations "good", i.e. are they close to the true value?

2.: Higher Order Approximation. Compute the n-th order Taylor Approximation to the exponential function for $x_0 = 0$ for variable $n \in \mathbb{N}$. Can you find an infinite sum representation for the exponential function using polynomial terms?

b.) Matrix Functions (online)

Consider a matrix $A \in \mathbb{R}^{n \times n}$, $n \in \mathbb{N}$.

(i) Show that $\frac{d}{dx}(Ax) = A$.

(ii) What is the derivative of $f : \mathbb{R}^n \mapsto \mathbb{R}, x \mapsto x'Ax$?
Hint: Use *(i)* and the multivariate product rule.

(iii) If $A = \begin{pmatrix} 1 & \alpha \\ \beta & 4 \end{pmatrix}$, can you find values for α and β so that the second derivative of $x'Ax$ is positive definite everywhere? Can you find an alternative combination where A is positive semi-definite but not positive definite?