E600 Mathematics

Fall Semester 2021

Self-study Exercises 2

to be solved on Thursday, week 1

Topics: Metric and Norm, Basis, Matrix Algebra

Exercises for Chapter 1

Exercise 1: Testing for Set Properties, again (online)

Let $S_4 := \{x \in \mathbb{R}^2 : x_1 x_2 = 5\}$. Is this set open, closed, compact and/or convex?
Hint: $f : \mathbb{R}^2 \mapsto \mathbb{R}, x = (x_1, x_2)' \mapsto x_1 x_2$ is continuous, and thus $\lim_{n\to\infty} x_1^n x_2^n = \lim_{n\to\infty} x_1^n \lim_{n\to\infty} x_2^n$.

Exercise 2: Basis of a Vector Space

a.) Bases of the $\mathbb{R}^2$

Which of the following sets are bases of the $\mathbb{R}^2$ (S&B, Ex. 11.12)?

$$S_1 = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-2\\-2\end{pmatrix}\right\}, \quad S_2 = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\-2\end{pmatrix}\right\}, \quad S_3 = \left\{\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-2\\2\end{pmatrix}\right\}, \quad S_4 = \left\{\begin{pmatrix}1\\-1\end{pmatrix}, e_2\right\}$$

b.) A Basis of a Function Space

Consider the set of second order polynomials,

$$\mathbb{P}_2(X) := \{f : X \mapsto \mathbb{R} : (\exists a, b, c \in \mathbb{R} : f(x) = ax^2 + bx + c)\}$$

It turns out to be the case that $\mathbb{P}_2(X)$ is a subspace of $\mathbb{F}(X, \mathbb{R})$, the space of all functions mapping from X to $\mathbb{R}$. Find a basis for $\mathbb{P}_2(X)$, i.e. a set of functions $f_1, f_2, \dots, f_k$ such that (1) every second order polynomial can be written as a linear combination of these functions and (2) the functions are all linearly independent, i.e. they can not be written as linear combinations of each other.

Bonus Question: Show that $\mathbb{P}_2(X)$ is a subspace of $\mathbb{F}(X, \mathbb{R})$. What is its dimension?

Exercises for Chapter 2

Exercise 3: Matrix Multiplication

a.) Two Matrices (online)

Determine whether the following matrices exist, and if so, compute them: AB, $B'A'$ and BA for

$$A = \begin{pmatrix} 0 & 2 \\ 3 & -5 \\ -2 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \end{pmatrix}$$

Hint: Be aware of the rules for transposition and matrix operations to take some shortcuts!

b.) Some more Products

Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -4 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -5 & 3 \\ 2 & 4 \end{pmatrix}.$$

Determine whether the following matrices exist, and if so, compute them:

1. AB	2. BA	3. $B'A'$
4. $BA + C$	5. $AB + C$	6. $(AB + C)'$

Hint: Be aware of the rules for transposition and matrix operations to take some shortcuts!

c.) Right-Multiplication of Vectors and Dimensionality

Let A be the matrix as in b.). What $n \in \mathbb{N}$ must we choose so that $x \in \mathbb{R}^n$ can be right-multiplied to A, i.e. as Ax? What about $A'x$?

Exercise 4: Elementary Matrix Operations

Here, we convince ourselves again that the elementary operations really work in the way we introduced them: Consider the matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

Define the matrix E so as to

1. interchange rows 2 and 3 (call the matrix E_1),
2. multiply rows 1 and 3 with $\lambda = 5 \neq 0$ (call the matrix E_2),
3. subtract two times row 1 from row 2 (call the matrix E_3).

Multiply out EA for E_3 and check that indeed, the respective operation is performed.

Exercise 5: Determinant, Definiteness and Eigenvalues

a.) Determinant Rules (online)

For the following matrices, compute the determinant using an appropriate rule.

1. $A = \begin{pmatrix} 3 & 8 \\ 2 & -1 \end{pmatrix}$

2. $B = \begin{pmatrix} 1 & -2 & 4 \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{8} \\ 1 & 2 & 1 \end{pmatrix}$

3. $C = \begin{pmatrix} 0 & 0 & 2 \\ 3 & 1 & -1 \\ 2 & 2 & 4 \end{pmatrix}$

Hint: You can test your understanding of the Laplace method by using an appropriate expansion at 3. (of course, the 3×3 rule is still perfectly fine here as well).

b.) Definiteness (online)

Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

Is A positive or negative (semi-)definite? What can you say about the invertability of A from this fact?

Exercise 6: Matrix Inversion

a.) Concrete Examples

For the following matrices, perform a test for invertability (using e.g. the determinant) and, if possible, compute the inverse matrix, using either a shortcut theorem or the Gauss-Jordan method:

1. (online) $A = \begin{pmatrix} 3 & 8 \\ 2 & -1 \end{pmatrix}$

2. (online) $B = \begin{pmatrix} -3 & 2 & 4 \\ -6 & 5 & 4 \\ 1 & -1 & 0 \end{pmatrix}$

3. $C = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

b.) The 2x2-Rule

(i) Derive 2x2 rule for the inverse, i.e. show that

$$\text{If } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } \det(A) = ad - bc \neq 0, \text{ then } A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

To do so, either use the Gauss-Jordan algorithm or multiply out AA^{-1} and $A^{-1}A$.

(ii) Can you invert C of Problem 2? If so, what is C^{-1}?

Bonus Question: A Huge Matrix (online)

In rare cases, you may come across applications where you need to invert "bigger" matrices than we usually deal with, i.e. matrices of higher dimension. Generally, this is quite computation-intensive, however, some matrices make your life easier than others. This exercise gives you some examples where despite matrix size, invertability checks and inversion are still manageable.

For the following matrix, perform a test for invertability (using e.g. the determinant) and, if possible, compute the inverse matrix, using either a shortcut theorem or the Gauss-Jordan method:

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 5 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{pmatrix}$$

Exercise 7: Eigenvalues, Definiteness and Invertability

Let $A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$. Determine the eigenvalues of A. What can you say about A's definiteness? Is A invertible? How could you have checked invertability more directly?