E600 Mathematics

Fall Semester 2021

Self-study Exercises 1

to be solved on Tuesday, August 24

Topics: Mathematical Fundamentals, Metric and Norm, Set Properties

This is the first set of self-study exercises for the class of 2021. These exercises are meant to be worked on by you, individually or in groups, on the days where there is no class. Doing so is highly advised as a great deal of mathematical knowledge comes only through application and practice. However, there is no need to hand in solutions, and also no punishment if you don't find the time to work on (all of) them. Especially, not completing the exercises does also *not* mean that you "drop out" of the class or will be unable to follow future lectures.
Some exercises are marked with "online". This means that they are taken from the respective chapter's collection of exercises on the webpage of the course (`e600.uni-mannheim.de`), where you can also find the solutions. Of course, you should try to solve the problems first without looking at the solution, but for these exercises, you can check if your approach works or what you're missing, which will help for the remaining problems.
The solutions are discussed in class on the next day – if we are short on time, some online problems might be left out. That being said, your first goal in working on these exercises is understanding what the problem is and developing a rough idea of how to approach it. Doing so will already help a lot as this will make our discussions of the solutions much more accessible. Thus, if you can solve the exercises, all the better, but do not feel frustrated if you can't!

Exercises for Chapter 0

Exercise 1: Notation and Logic

a.) Validity of Arguments

Consider an argument with structure "Premise 1 and Premise 2 imply Conclusion". Is the argument valid for the given combinations of premises and conclusion?
Hint 1: Validity is given only if the premises *necessarily* imply the conclusion, it does not suffice if the premises do not contradict the conclusion.
Hint 2: Recall that you can use circles to illustrate mathematical arguments.

Nr.	Premise 1	Premise 2	Conclusion
1	All dogs do not meow	Snoopy is a dog	Snoopy does not meow
2	All cats dislike rain	Snoopy dislikes rain	Snoopy is a cat
3	A free person has nothing to lose	A prisoner is not a free person	A prisoner has something to lose
4	If it rains, we don't play outside	We play outside	It's not raining
5	For all $x \in S$, if $x > 1$, then $x > 2$	$0 \in S$ and $0 > 1$	$0 > 2$

Is 5. sound if $S \subseteq \mathbb{R}$, and the ">" relation is defined in the usual way?

b.) Arguments and Sets

1. Define some appropriate notation and write down Nr. 1 of 1.a) as a set statement.
2. Draw Nr. 2 of 1.a.) using the "circle approach" to sets.

c.) Quantifiers and Implication (online)

Assess whether the following statements are necessary, sufficient, equivalent, or neither of the previous, for $S := (\forall x \in A : (x - \pi \in \mathbb{Z}))$. You may assume that A is not the empty set, so that it contains at least one element.

1. $\exists x \in A : (x - \pi \in \mathbb{Z})$
2. $\forall x \in A : (x - \pi \in \mathbb{N})$
3. $\exists! x \in A : (x - \pi \in \mathbb{Z})$
4. $\nexists x \in A : (x - \pi \notin \mathbb{Z})$
5. $\nexists x \in A : (x - \pi \notin \{1, 2, 3\})$
6. $A = \{1, 2, 3\}$
7. $A = \{1 + \pi, -1 + \pi\}$
8. $\forall x \in A : (x - 4 \geq 2)$

Exercise 2: Set Theory

a.) Set Operations (online)

Compute union, intersection and both set differences for $A = \{1, 3, 5, 7, 9\}$ and $B = \{-1, 0, 1, 2, 3, 4, 5\}$.

b.) Statements related to Sets

Let $A = \{2,4,6,8,10\}$ and $B = \{1,3,5,7,9\}$. Which of the following statements are true?

1. $2 \in A$
2. $3 \ni B$
3. $4 \notin B$
4. $A \in \mathbb{N}$
5. $A = \{2n : n \in \mathbb{N}\backslash\{0\}\}$
6. $A \cup B = \mathbb{N}$
7. $A \cup B \subset \mathbb{N}$
8. $A = \{2,4,6,8,10,2,4,6,8,10\}$
9. $A = \{2,4,6,8,10,\{2,4,6,8,10\}\}$
10. $B = \{n \in \mathbb{N} : ((\exists m \in \mathbb{N} : n = 2m+1) \vee n < 10)\}$
11. $B = \{n \in \mathbb{N} : ((\exists m \in \mathbb{N} : n = 2m+1) \wedge n < 10)\}$
12. $A = [2,10) \cap \mathbb{N}$

For the first and last statement, if they are false, can you modify them to make them true?

Exercise 3: Functions

a.) Codomain and Range

Give an example for a function f for which the codomain is not equal to the range of f.

b.) Image of a Set under a Function

Let $f : X \to Y$ be a function, and let $A \subset X$. If we say that y is an element of $f[A]$, i.e. $y \in f[A]$ what exactly do we know about y?
A. $f(y) \in A$.
B. $f^{-1}(y) \in A$.
C. $y \in X$.
D. For some $x \in A$, it holds that $f(x) = y$.
E. $y \in A$.

c.) Preimage of a Set under a Function

Let $f : X \to Y$ be a function, and let $B \subset Y$. If we say that x is an element of $f^{-1}[B]$, i.e. $x \in f^{-1}[B]$, what exactly do we know about f and x?
A. $f(x) \in B$
B. $\exists y \in B : x = f^{-1}(y)$
C. $x \in B$
D. $f(x) = B$
E. f is invertible.
F. $f^{-1}(B) = x$

d.) Derivative using the Appropriate Rule

Calculate $f'(x)$ for $f : \mathbb{R} \mapsto \mathbb{R}, x \mapsto \sin((2x+4)^2)$.

Exercises for Chapter 1

Exercise 4: Key Concepts of Vector Spaces

a.) The Scalar Product (online)

Compute the scalar product of v and w when...

1. $v = (2,3)'$ and $w = (-2,9)'$
2. $v = (3,2,\ln(8),-6)'$ and $w = (\ln(2),1,-1,1/4)'$ (*Hint:* $a\ln(b) = \ln(b^a)$)
3. $v = (4,19)'$ and $w = (2,3,5)'$

b.) More Scalar Products

Compute $a \cdot b$ for

1. $a = (2,5,1)', b = (1,1,3)' \in \mathbb{R}^3$
2. $a = (2,0,-3,4), b = (9,-8,7,-6) \in \mathbb{R}^4$

What do you tell your colleague who claims to have found $v \in \mathbb{R}^n$ so that $v \cdot v = -1$?

c.) The Binary Metric

Consider a real vector space $\mathbb{X} = (X,+,\cdot)$, and define the binary metric

$$d_B : X \times X \mapsto \mathbb{R}, d_B(x,y) = \mathbb{1}[x \neq y]$$

Show that the function indeed constitutes a metric, i.e. show that it satisfies the three properties that define a metric function.

In case you are not familiar with the notation used here, $\mathbb{1}[S(x)]$ is a so-called indicator function for a statement $S(x)$ related to x that takes the value 1 when $S(x)$ is true and 0 otherwise. Accordingly,

$$d_B(x,y) = \mathbb{1}[x \neq y] = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

d.) A Norm (online)

The course discusses the most common examples of norm functions as used in practice. However, there are of course a variety of other functions that are norms. One such example is the following function on $\mathbb{R}^3$:

$$n : \mathbb{R}^3 \mapsto \mathbb{R}, x = (x_1, x_2, x_3)' \mapsto |x_1| + \max\{2|x_2|, 3|x_3|\}.$$

Verify that this function constitutes a norm according to our definition by establishing the three norm properties for n.

e.) Norm Equivalence (online)

A key concept related to norms that the course does not discuss is *norm equivalence*. You may recall that we said that a function may be continuous in one metric space but not in the other, depending on the metric chosen. Similarly, convergence of sequences may depend on the metric chosen. A further appealing aspect of norm-induced metrics over general metrics is that at least in metric spaces over $\mathbb{R}^n$ with finite $n \in \mathbb{N}$, all of our usual p-norms are *equivalent*, which means that if a function f satisfies continuity or a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges with respect to one norm, they do so with respect to the other as well.

It turns out to be sufficient for norm equivalence that we can reduce the differences in two norms to some constants that matter little in the ε/δ-arguments we use to investigate convergence, continuity and related concepts. Formally, two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on X are said to be equivalent if there exist constants $0 < c \leq C < \infty$ such that

$$\forall x \in X : c\|x\|_a \leq \|x\|_b \leq C\|x\|_a.$$

Show that all p-norms are equivalent on any $\mathbb{R}^n$, i.e. that $\forall p, q \in \mathbb{N} \cup \{\infty\}$, $\|\cdot\|_p$ is equivalent to $\|\cdot\|_q$ on $\mathbb{R}^n$ for any $n \in \mathbb{N}$. (*Hint:* this is easiest done in two steps: (1) show that any p-norm is equivalent to the maximum norm, and (2) show that if a p- and q-norm are both equivalent to the maximum norm, the p- and q-norm are also equivalent to each other.)

Exercise 5: Testing for Set Properties

In this exercise, we practice the investigation of key set properties discussed in lecture 1. As set properties refer to metric or normed vector spaces, respectively, you need to know which distance measure to consider. You can assume (as always when nothing else is explicitly specified) that we deal with Euclidean spaces, but feel free to choose any other p-norm (including $p = \infty$) that may work better for you (recall the previous exercise: p-norms are equivalent, it does not matter which one you choose).

Remark: The numbering of sets is intentional; S_2 was skipped on purpose so as to keep the notation of the online solutions.

a.) A subset of the real line (online)

Let $S_1 := \{x \in \mathbb{R} : x^2 \leq 4\}$. Is this set open, closed, compact and/or convex?

b.) Two dimensions (online)

Let $S_3 := \{x \in \mathbb{R}^2 : x_1 \leq 3\}$. Is this set open, closed, compact and/or convex?