E600 Mathematics

Fall Semester 2021

In-class Exercises for Chapter 4

Discussed in class on Thursday, week 2

Topics: Optimization

Problem 1: Solution Existence for Univariate Concave Functions

Consider a univariate, real-valued function $f:(\underline{x},\bar{x})\mapsto\mathbb{R}$, $\underline{x},\bar{x}\in\mathbb{R}$ so that $\underline{x}<\bar{x}$. Suppose that
(i) f is once differentiable,
(ii) f is concave, and that
(iii) there exist $a,b,c\in(\underline{x},\bar{x})$ with $a<b<c$ so that $f(a)<f(c)<f(b)$.
Can you argue that f assumes a global maximum on $(\underline{x},\bar{x})$?

Hint 1. Recall that concavity is a desirable feature in unconstrained maximization.
Hint 2. Think about combining the Mean Value and Intermediate Value Theorem.

Problem 2: Unconstr. Optimization and Matrix Definiteness (online)

In the exercises of Chapter 3, we investigated definiteness of the second derivative of $x'Ax$ for $A=\begin{pmatrix}1 & \alpha\\ \beta & 4\end{pmatrix}$. Recall that the second derivative was $A+A'$ and that for $v\in\mathbb{R}^2$,

$$v'(A+A')v=2(v_1^2+(\alpha+\beta)v_1v_2+4v_2^2)$$

(i) Can you use an optimization problem approach to find the values for α and β where $A+A'$ is not positive semi-definite, i.e. where there exist $v\in\mathbb{R}^2$ for which $v'(A+A')v<0$?

To help you with the solution, note that if we write $v=\lambda v_0$, we have

$$v'(A+A')v=2\lambda^2(v_{0,1}^2+(\alpha+\beta)v_{0,1}v_{0,2}+4v_{0,2}^2)$$

Hence, the magnitude does not matter for the property of being strictly negative, and you can reduce the search for v with $v'(A+A')v<0$ to a direction vector. Because the magnitude of the direction vector does not matter and as if $v_1=0$, then the expression is weakly positive, there is no loss in setting $v_{0,1}=1$ and restricting the search to the value $v_{0,2}$ that yields $v'(A+A')v<0$.

Problem 3: Saving Time in Optimization

Solve

$$\max \frac{4}{3}x^2 + y + xz \quad \text{s.t.} \quad \|(x,y,z)\|_2 \leq 1$$

where $\|\cdot\|_2$ is the Euclidean norm of the $\mathbb{R}^3$.

You will get a multitude solutions to the FOC. For any one of these, using the second order condition, you would have to check two determinants of the Bordered Hessian, one for a 3×3 and one for a 4×4 matrix. What may help to avoid this is that you can make an argument for solution existence, and also the Lagrangian multiplier condition.

Three simplifications and intuitions can be extremely helpful:

- If $\|(x,y,z)\|_2 < 1$, we can increase the objective varying marginally y.
- By norm non-negativity, $\|(x,y,z)\|_2 \leq 1$ is equivalent to $\|(x,y,z)\|_2^2 \leq 1^2$.
- Closed balls of the $\mathbb{R}^n$ are compact.