E600 Mathematics

Fall Semester 2021

In-class Exercises for Chapter 3

Discussed in class on Tuesday, week 2

Topics: Functional Analysis, Multivariate Differentiation

Problem 1: Intermediate Value Theorem (online)

Here, we consider a new theorem that the lecture had not introduced:

Theorem 1. ***(Intermediate Value Theorem)*** *Let $f : X \mapsto \mathbb{R}$ for some set $X \subseteq \mathbb{R}$, and assume that f is continuous. Then, for any $a, b \in X$ with $a < b$ and $f(a) \leq f(b)$ (and $f(a) \geq f(b)$), for any $y \in [f(a), f(b)]$ (for any $y \in [f(b), f(a)]$), there exists $c \in [a, b]$ with $f(c) = y$.*

Verbally, this theorem relates to the intuition of being able to draw continuous functions without lifting the pen: if the continuous function attains two different values within the codomain, it will also reach every value in between along the way. The exercise to follow extends this intuition by establishing that for two continuous functions, if one lies above the other at one point but below at another point, then the functions must intersect in between the points.

a.) Intersecting Continuous Functions

Use the intermediate value theorem to show that if two continuous functions f, g with domain $X \subseteq \mathbb{R}$ and codomain $\mathbb{R}$ are such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$ for some $a, b \in X$, then there exists a value $x \in X$ in between a and b (i.e., $x \in [a, b]$ when $a \leq b$ and $x \in [b, a]$ else) such that $f(x) = g(x)$.

b.) Surjectivity

Consider the function $f : \mathbb{R} \mapsto \mathbb{R}, x \mapsto \sin(x) - \frac{3}{2}x$. Is this function surjective? Is it invertible?
Hint: Use the result of a.). We know about injectivity from the self-study exercises.

Problem 2: Mean Value Theorem

Prove the Mean Value Theorem, i.e. show that for $f \in D^1(X)$, $X \subseteq \mathbb{R}$, for any $a, b \in X$ so that $a < b$, there exists $x_0 \in (a, b)$ so that

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

What does this imply for the existence of "critical values" of f on X, i.e. points $x \in X$ where $f'(x) = 0$? Illustrate this idea graphically.

Problem 3: Multivariate Chain Rule

To be skipped if we are short on time.

a.) Deriving a more familiar Expression

Let $f : Y \times Z \mapsto \mathbb{R}$, $X = Y \times Z \subseteq \mathbb{R}^n$, i.e. consider a function f of the form $f(x) = f(y, z)$ where y and z are potentially vectors. Further, define $g(z) = f(y(z), z)$, so that we vary y in a specific fashion related to z. Using the multivariate chain rule, derive that

$$\frac{dg}{dz} = \frac{\partial f}{\partial y}\frac{dy}{dz} + \frac{\partial f}{\partial z}$$

or respectively, that for any $z \in Z$,

$$\frac{dg}{dz}(z) = \frac{\partial f}{\partial y}(y(z), z)\frac{dy}{dz}(z) + \frac{\partial f}{\partial z}(y(z), z)$$

b.) Application

Use either version of the multivariate chain rule to derive the marginal indirect utility of consumption for x_2 when $u(x_1, x_2) = \sqrt{x_1 x_2}$ and the budget constraint is $x_1 + 2x_2 = 9$.