E600 Mathematics

Fall Semester 2021

In-class Exercises for Chapter 2

Discussed in class on Friday, week 1

Topics: Matrix Algebra, Equation Systems

Problem 1: Laplace Expansion (online)

In the course, we said that we typically deal with matrices of manageable size when computing the determinant, or that the matrix has a convenient structure (triangular, diagonal), where computing the determinant is as simple as multiplying all diagonal elements to obtain the trace. In some applications, however, you may not be that lucky, and revert to the general Laplace rule. Especially if there are zeros in the matrix, this method is still quite easily and quickly applied; this exercise is supposed to convince you of this fact.

Compute $\det(A)$ when

$$A = \begin{pmatrix} 1 & 3 & 0 & 0 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ 0 & 12\cdot\pi & 3 & -5 & 1 \\ 0 & 2 & 0 & -1 & 1 \\ 1 & 1 & 0 & 2 & 3 \end{pmatrix}$$

Problem 2: The Nullspace and the Dimension of the Solution Set

A key concept related to solving equation systems in matrix notation that we haven't touched in the lecture is the "Nullspace" of a matrix A, also called the kernel, defined as

$$\ker A = \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}.$$

It is straightforward to verify the subspace property since $A(\lambda x + \mu y) = \lambda Ax + \mu Ay$. Here, we deal with its relation to the set of solutions. It will allow us to more formally address our intuition of free variables through the fundamental theorem of Linear Algebra, a really powerful result that you should have seen at least once!

a.) Solutions and the Kernel

Show that if there exists a solution x^* to the equation system $Ax = b$ in matrix notation, then x^s is a solution if and only if there exists an $x_0 \in \ker A$ so that $x^s = x^* + x_0$.

Also answer the following:

1. What can you conclude for the dimension of the number of free dimensions in the problem?

2. Suppose that $B_K(A) = \{b_{K,1}, \ldots, b_{K,d}\}$, $d \in \{0, 1, \ldots, m\}$, is a basis of ker A. How can you use $B_K(A)$ to represent the solutions of $Ax = b$?

b.) The Fundamental Theorem of Linear Algebra

The theorem tells us about the interrelation of the number of free dimensions and the rank: it states that for $A \in \mathbb{R}^{n \times m}$,

$$\dim(\ker A) = m - \text{rk } A.$$

Note that unique solutions to $Ax = b$ can only exist if $\dim(\ker A) = 0$ (recall a.)), which already rules out unique solutions if $n < m$, i.e. strictly more unknowns than equations.

Noting that $(1, 1, 1)'$ is a solution, apply the theorem to determine the number of free variables in $Ax = b$ when

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

and use a.) to characterize the set of solutions.