In-class Exercises for Chapter 1

Discussed in class on Wednesday, week 1

Topics: Vector Spaces, Basis and Norms

Problem 1: Subspaces, Linear Dependence and Basis

a.) A Proper Subspace of $\mathbb{R}^3$

Prove that

$$S_2 := \{x = (x_1, x_2, x_3)' \in \mathbb{R}^3 : x_2 = 0\}$$

gives rise to a proper subspace of $\mathbb{R}^3$. What is its dimension?
Hint: use the linear combination definition of the subspace.

b.) Bases

Think of two different bases for $\mathbb{R}^3$. Include $b_1 = (1,1,0)'$ and $b_2 = (1,0,4)$ in the second.

Problem 2: Norm and Metric in Vector Spaces

a.) The Norms we use are actually Norms

Recall the most commonly used norms on $\mathbb{R}^2$:

- 1-norm ("Manhattan"): $\|x\|_1 = |x_1| + |x_2|$
- 2-norm ("Euclidean"): $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$
- infinity-norm ("Maximum"): $\|x\|_\infty = \max\{|x_1|, |x_2|\}$

(i) Show that the Euclidean norm constitutes a norm.
Hint: You may use the **Cauchy-Schwarz inequality** for the Euclidean space $(\mathbb{R}^n, \|\cdot\|_2)$, which states that for any $x, y \in \mathbb{R}^n$:

$$|x \cdot y| \leq \|x\|_2 \|y\|_2.$$

(ii) Except for the triangle inequality, the norm property proofs for the other norms are highly analogous. To convince yourself that the Maximum norm is also a norm, show that it satisfies the triangle inequality.

(iii) Sketch the unit-closed ball of these norms, i.e. $\bar{B}_1(0) = \{x \in \mathbb{R}^2 : \|x\| \le 1\}$

Remark: The arguments establishing that the norms we considered here constitute norms on any $\mathbb{R}^n$, $n \in \mathbb{N}$ proceed in perfect analogy to the solutions of this exercise.

b.) Inverse Triangle Inequality

Let $(\mathbb{X}, \|\cdot\|)$ be a normed vector space. Show the inverse triangle inequality, that is, prove that

$$\forall x, y \in \mathbb{X} : \|x - y\| \ge \big| \|x\| - \|y\| \big|.$$

c.) Norm Continuity

Show that any norm is continuous. More formally: show that if $\mathbb{X} = (X, +, \cdot)$ is a vector space and $\|\cdot\|$ defines a norm on X, then it holds that for any $x_0 \in X$,

$$\forall \varepsilon > 0 \exists \delta > 0 : (x \in B_\delta(x_0) \Rightarrow \|x\| \in B_\varepsilon(\|x_0\|))$$

or equivalently

$$\forall \varepsilon > 0 \exists \delta > 0 : (\|x - y\| < \delta \Rightarrow \big| \|x\| - \|y\| \big| < \varepsilon).$$