E600 Mathematics

Welcome and Chapter 0: Fundamentals of Mathematics

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Welcome to E600 Mathematics!

Overview

- What is E600 Mathematics?
  - Introductory module on mathematical background of economists
  - Course is not mandatory, **no exam**
    - $\Rightarrow$ All efforts are a voluntary, but rewarding investment into your MSc!
  - more detail in a minute...
- Instructor: Martin Reinhard
  - PhD student at CDSE
  - Email: mareinhard@mail.uni-mannheim.de
  - no scheduled office hours, individual meetings can be arranged

- Main Course Objectives
  - Discuss and practice mathematical concepts central for economics
  - Get more familiar and comfortable with logic and notation
  - Know and be able to apply main theorems
  - Have a rough idea of why they are true (because "why" is sufficient for "how")
- Why care?
  - Mannheim has a strong quantitative orientation
  - Logical reasoning and mathematical methods are central to economics

  $\Rightarrow$ Make your life a bit easier in the years to come

1. Logistics and Organization

What: Content

0. Fundamentals of Mathematics (Logic, Functions, Sets)
1. Vector Spaces
   - Metric and norm: mathematical distances
   - Set properties: bounded, open/closed, compact, etc.
2. Matrix Algebra
   - math with matrices: addition, multiplication, inversion
   - linear equation systems
3. Differential Calculus
   - Computing derivatives and integrals...
   - when functions have vectors as inputs/outputs
4. Optimization
   - Unconstrained and constrained
   - Single and multiple variable problems

1. Logistics and Organization

When?

<table>
<tr><th>Time</th><th>Mon, Aug 23</th><th>Wed, Aug 25</th><th>Fri, Aug 27</th><th>Tue, Aug 31</th><th>Thu, Sep 2</th></tr>
<tr><td rowspan="2">9:00 - 10:30</td><td>Welcome (Administration)</td><td rowspan="2">Lecture Ch. 1</td><td rowspan="2">Lecture Ch. 2</td><td rowspan="2">Lecture Ch. 3</td><td rowspan="3">Lecture Ch. 4</td></tr>
<tr><td>Course Logistics</td></tr>
<tr><td rowspan="2">10:45 - 12:15</td><td>Review of Ch. 0</td><td rowspan="2">Solutions Self-Study 1</td><td rowspan="2">Solutions Self-Study 2</td><td rowspan="2">Solutions Self-Study 3</td></tr>
<tr><td>In-class Exercises Ch. 0</td><td rowspan="2">Solutions Self-Study 4</td></tr>
<tr><td rowspan="2">13:15 - 14:45</td><td rowspan="4">Lecture Ch. 1</td><td>In-class Exercises Ch. 1</td><td>In-class Exercises Ch. 2</td><td>In-class Exercises Ch. 3</td></tr>
<tr><td rowspan="3">Lecture Ch. 2</td><td rowspan="3">Lecture Ch. 3</td><td rowspan="3">Lecture Ch. 4</td><td>In-class Exercises Ch. 4</td></tr>
<tr><td rowspan="2">15:00 - 16:30</td><td rowspan="2">No class/exercise</td></tr>
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- $\approx$ 4 blocks per chapter
- 60% lecture, 40% exercise
- Days without class: self-study exercises
- Schedule is broad outline, may take more/less time according to demand

1. Logistics and Organization

How? 1/2

- Web: `e600.uni-mannheim.de`
  - Full text for all chapters as Online Course
    - Interactive design: short questions, review quizzes
    - Focused on core competencies: few proofs, involved concepts left out
  - Additional practice problems for each chapter
- Course Material (available at `e600.uni-mannheim.de/downloads`)
  - Self-study and in-class exercise sheets, slides
  - Companion script: everything we do in class (and more) + all proofs
- Online Course & Script
  - *comprehensive* resources: meant as your MSc companion to consult for any issue related to mathematics
  - *self-sufficient* resources: consistent notation, rely only on known/previously introduced concepts

- Self-study
  - Online course + script, online exercises
  - Work through chapters/topics selectively & at your own pace
- Attend class
  - Class-based module “forces” you to spend time with the material
  - I can help you to read between the lines
  - Don’t hesitate to **ask questions!** (also after class)
  - Learn how to come up with solutions in the exercise sessions
- Work on self-study (“homework”) problems

1. Logistics and Organization

Self-study Exercises

- Days without class: review and work on self-study exercises
  - Expectation: understand all problems, solve the easier ones
  - No need to hand in anything
  - Solutions discussed in class on the next day
- Work in groups
  - Get to know your fellow students (better)
  - Divide problems among group members, explain solutions to each other
  - Do you want/need a group?
- Please also take the time to fill in a short feedback survey

- E600 has **a lot of material**
  - many important concepts to cover in little time
  - not all is new; you need not know everything by the end of this class
- After this class, you should. . .
  - be less afraid of mathematical notation and formal arguments
  - have heard of the math. concepts central to economics and know where to look them up
  - everything else is a plus!
- In times of COVID: actively target your involvement in this class!
  - do not hesitate ask questions during and after class
  - work in groups
  - give feedback through the website

1. Logistics and Organization

Expectation: An Example

- In 5 weeks, you are sitting on a problem set for Microeconomics A.
- The exercise asks you: "Show that the budget constraint given above defines a bounded set."
- Because you followed this class well, you will think:
  - X "okay, I know exactly how it works, this should be easy!"
  - ✓ "I know intuitively what a bounded set is, and why it's helpful if we can show that the budget set is bounded. Let me quickly
    - look up the definition of a bounded set, and
    - check how I can go about showing boundedness."

Chapter 0: Fundamentals of Mathematics

2. Fundamentals: Why Math?

aka "Why not a review of Micro/Macro/Econometrics?"

- Economists (among many other professions) put great emphasis on logical, rational reasoning – math is the purest form of this
- Economic investigations: mathematical or statistical models
- Math is actually less messy than the real world
  - No conceptual ambiguity: "fair tax system" vs. "Derivative of a given function f"? (existence?)
  - Full information: statements can be proved/disproved with certainty

$\Rightarrow$ "lab conditions" to test logical reasoning

- Math = "language" very similar to normal English
  - Vocabulary: words + notation; sentence = "statement"
  - Most important "sentences": quantifiers
    - Existential quantifier $\exists \ldots$
    - Universal quantifier $\forall \ldots$
    - Example: $\forall n \in \mathbb{N} : n \geq 0$
  - Grammar: rules that determine whether statements are meaningful
- Argument: statement about relationship of statements ("if ..., then")
  - English: "Any animal that is not a cat or a dog is a horse."
  - Math: $\forall a \in A : ((a \notin C \wedge a \notin D) \Rightarrow a \in H)$

- Particularities of the mathematical language
  - The logical "or" ($\vee$) reads as "and/or"
  - "greater than" includes equality, $>$ reads as "strictly greater than"
- Statements: meaningful (grammar) and true (content)?

Bananas blue	$\forall x \in \mathbb{R} : (\forall y \in \mathbb{R} : x \in y)$
Bananas are blue	$\forall x \in \mathbb{R} : (\forall y \in \mathbb{R} : x > y)$
Bananas are yellow	$\forall x \in \mathbb{R} : (\exists y \in \mathbb{R} : x > y)$

- Negation: usually more direct when re-written
  - E.g.: $\neg(x \in A) \Leftrightarrow x \notin A$
  - Quantifiers: $\neg(\forall \ldots : S) \Leftrightarrow (\exists \ldots : \neg S)$ and vice versa (cf. exercises)

4. Fundamentals: Arguments and Reasoning

Vocab and Grammar of Mathematics 3/3

- Classic argument: 2 premises imply one conclusion
  - (P1: All teachers like coffee, P2: I am a teacher $\Rightarrow$ C: I like coffee)
- More generally: may have multiple premises/conclusions, *equivalence*
- Argument: valid and sound?
  - valid: asserted relationship is correct
  - sound: valid argument and the premises are true
  - Ex. 1: (Mannheim is a German city $\wedge$ Germany is an Asian country) $\Rightarrow$ Mannheim is an Asian city
  - Ex. 2: (Mannheim is a German city $\wedge$ Germany is a European country) $\Rightarrow$ Mannheim is a European city
- Mathematical proof: establish non-obvious validity of argument
- Caution: arguments do not address causation (e.g. coffee)!

- Consider a statement/"event" E
  - Sufficient conditions S *imply* E: S guarantees E ("if")
  - Necessary conditions N *are implied by* E: $\neg N$ rules out E ("only if")
  - Necessary *and* sufficient = equivalent ("if and only if")
- Example: $E := (x \in \mathbb{N})$
  - sufficient: $x = 2$
  - necessary: $x \in \mathbb{Z}$
  - equivalent: $(x \in \mathbb{Z} \wedge x \geq 0)$

5. Fundamentals: Sets

- Set $=$ collection of distinct objects ("elements")
  - $\{1, 2, \pi\}$ vs. $\{1, 1, \pi\}$ vs. $\{n \in \mathbb{N} : n > 10\}$
  - Elements can be sets, matrices, functions, whatever
  - Key concept: subset $A \subseteq S \Leftrightarrow (x \in A \Rightarrow x \in S)$; proper subset $A \subset S$
- Intervals: $I \subseteq \mathbb{R}$, write $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
  - (right-/left-) closed and open interval: $(a, b]$, $[a, b)$, $[a, b]$
- Set relations: equal vs. disjoint, complement
- Operations
  - Union $A \cup B$, Intersection $A \cap B$, Set difference $A \backslash B$ (circles)
  - Power set $\mathcal{P}(A) = \{S \subseteq X : S \subseteq A\}$: set of subsets
  - Index sets for *sets/sequences of sets*: $\bigcup_{i \in I} A_i$, $\bigcap_{i \in I} A_i$ (e.g. $I = \mathbb{N}$; what does pairwise disjoint mean?)

- Functions define a relation between sets
  - Usual notation: $f : X \mapsto Y, x \mapsto y = f(x)$
    - X: domain, Y: codomain
    - Mapping rule "$x \mapsto y = f(x)$" relates each $x \in X$ to exactly one $y \in Y$
  - Relation = collection of pairs $(x, y) = (x, f(x))$ in $X \times Y$:

$$G(f) = \{(x, y) \in X \times Y : y = f(x)\} = \{(x, f(x)) : x \in X\}$$

  - The *set* $G(f)$ is called the graph of f
- Concepts:
  - f is a function, "$f(x)$" is not!
  - Relations are sets

- Consider $f : X \mapsto Y$ (e.g. $f : \mathbb{R}_+ \mapsto \mathbb{R}, x \mapsto \sqrt{x}$)
  - Image of $A \subseteq X$, preimage of $B \subseteq Y$ ($A = [1, 4]$, $B = [0, 3)$)
  - $g : Y \mapsto Z$; composition $g \circ f$
  - Inverse function f^{-1} of f: $f^{-1} \circ f = f \circ f^{-1} = Id$
    - Id: identity function for which $Id(x) = x$
    - Existence depends on *bijectivity*
- The preimage and the inverse function are **fundamentally different objects**!

7. Fundamentals: Limits and Continuity in $\mathbb{R}$

- Sequence: $\lim_{n\to\infty} x_n = x \Leftrightarrow |x_n - x| \overset{n\to\infty}{\to} 0$
- Function $f : X \mapsto \mathbb{R}$ where $X \subseteq \mathbb{R}$:

$$\lim_{x\to a} f(x) = f_a \in \mathbb{R} \quad \Leftrightarrow \quad |f(x) - f_a| \underset{x\neq a}{\overset{x\to a}{\to}} 0$$

- Notational conventions
  - Divergence $\lim_{x\to a} f(x) = \pm\infty$ (e.g. $\lim_{x\to 0} \frac{1}{|x|} = \infty$)
  - Asymptotic behavior: $\lim_{x\to\infty} f(x) = c$: $|f(x) - c| < \varepsilon$ for a $x > C(\varepsilon)$
- Continuity of f at $a \in X$: $\lim_{x\to a} f(x) = f(a)$
  - Examples for discontinuity: $f(x) = \mathbb{1}[x > 0]$, $f(x) = \mathbb{1}[x = 0]$ (draw!)
  - Characterization: left and right limit exist at a and are equal to $f(a)$
  - Prove continuity: formal definition, left/right limits
  - Disprove: left $\neq$ right limit, sequence $a_n \to a$ where $f(a_n) \not\to f(a)$

- lim ... can be pulled into any continuous expression (including sums, products)
- Differentiable Functions: L'Hôpital's rule
  - e.g. $\lim_{x \to 0} \frac{\sin(x)}{x}$
- Sequences: Sandwich Theorem. Suppose $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \bar{x}$ and

$$\forall n \geq N \in \mathbb{N} : y_n \leq x_n \leq z_n.$$

  Then, $\lim_{n\to\infty} x_n = \bar{x}$.
  - e.g. $\lim_{n\to\infty} \frac{1}{n\cdot(\cos(n)+\mathbb{1}[n \text{ is even}]+2)}$

- Recall: proofs establish non-obvious validity of mathematical statement/fact
  - understanding them may improve your understanding of the fact
  - some are tedious (relative to the importance of the fact)
- How do we proceed?
  - The companion script gives all proofs in consistent notation
  - in class: easier proofs for the most central results
  - also: examples of methods of proof so that you have seen them once
    - Proof by induction
    - Equivalence proof in two steps
    - Contrapositive method of proof
    - ...