E600 Mathematics

Fall Semester 2020

Self-study Exercises 3

to be solved on the Weekend, September 19-20

Topics: Linear Independence Test, Multivariate Differentiation

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Exercise for Chapter 2

Exercise 1: Linear Independence Tests

The rank concept, and especially determination of the number of linearly independent vectors in a set, is an frequently perceived to be awkward to handle by students, perhaps because many courses do not teach an uniformly applicable method to determine/test for it, so that one has to revert to case-specific solution approaches, which are oftentimes non-obvious. Thus, you may gain a lot from being familiar with the matrix-based linear independence test, which motivates the exercise below.

Consider the following sets of vectors:

$$S_1 = \left\{ \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 13 \\ 37 \\ 16 \end{pmatrix} \right\}, \quad S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \\ 0 \end{pmatrix} \right\}, \quad S_3 = \left\{ \begin{pmatrix} -2 \\ 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 3 \\ 0 \end{pmatrix} \right\}.$$

are these sets linearly independent?

Hint: Recall that to perform the test, you need to bring the matrix of stacked column vectors to (generalized) triangular form and investigate the rank.

Exercises for Chapter 3

Exercise 2: Important Objects of Differential Calculus

Consider $x_0 \in X \subseteq \mathbb{R}^n$, $f : X \mapsto \mathbb{R}$. What is the type of the following objects:

$$\frac{\partial f}{\partial x_1}(x_0), \qquad \frac{df}{dx}, \qquad \frac{\partial}{\partial x_1}, \qquad \nabla f, \qquad \frac{df(x_0)}{dx}$$

Also answer the following:

1. Does $\frac{df}{dx_1}$ exist? What about $\frac{\partial f}{\partial x}$?
2. What is the difference between $\frac{df}{dx}(x_0)$ and $\nabla f(x_0)$?

Exercise 3: Invertability

a.) Some Examples

Determine which of the following functions are invertible, and if not, which criterion (injectivity or surjectivity) fails. Can you restrict domain and/or codomain to ensure invertability?[1]

1.	$f : \mathbb{N} \mapsto \mathbb{N},\ n \mapsto n^4$	2.	$f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto x^2$
3.	$f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto \cos(x)$	4.	$f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto 3$

b.) Monotonic Functions and Injectivity (online)

Show that if a function $f : X \mapsto \mathbb{R}$ where $X \subseteq \mathbb{R}$ is strictly monotonous, i.e. $\forall x, y \in X : (x > y \Rightarrow f(x) > f(y))$ or $\forall x, y \in X : (x > y \Rightarrow f(x) < f(y))$, then f is injective, i.e. that $\forall x, y \in X : (x \neq y \Rightarrow f(x) \neq f(y))$.

c.) Monotonic Functions and Injectivity: Application (online)

Consider the function $f : \mathbb{R} \mapsto \mathbb{R}, x \mapsto \sin(x) - \frac{3}{2}x$. Is this function injective?
Comment: We will investigate surjectivity in the in-class exercises for this function.

Exercise 4: Convexity

a.) Set Convexity: revisited (online)

Is the following set convex? Justify your answer!

$$S := B_\varepsilon(x_0) = \{x \in X : \|x - x_0\| < \varepsilon\}$$

for some $\varepsilon > 0$. What about a closed ball?

[1] We can always ensure surjectivity by restricting the codomain to $f[X]$. Here, let us focus only on more “crude” cuts, that is, restrictions to intervals.

b.) Hessian Criterion (online)

Investigate the following function with respect to (strict) convexity/concavity:

$$f : \mathbb{R}^2 \mapsto \mathbb{R}, x = (x_1, x_2)' \mapsto \exp(x_1) + x_1 x_2 + 5x_1 + 4$$

Hint: Recall that we can use the second derivative to investigate convexity.

c.) Norms (online)

Investigate the following function with respect to (strict) convexity/concavity:

$$f : \mathbb{R}^n \mapsto \mathbb{R}, x \mapsto \|x\|$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$, $n \in \mathbb{N}$.
Hint 1: We know that norms are continuous, but they need not be differentiable. Hence, the criterion for the second derivative is not useful here, and it is instructive to proceed with the "raw" definition of convexity.
Hint 2: The solution of a.) offers some insight into how we should approach this problem.
Hint 3: A function is only both convex and concave if it is linear, i.e. $f(x+y) = f(x) + f(y)$ for any possible arguments x, y. Thus, once you showed that f satisfies one of the properties, you can check the other by investigating whether f is/can potentially be linear.

d.) Support Restriction?

Use the Hessian Criterion to investigate whether the function

$$f(x,y) = \frac{1}{2}(y^3 + 2xy + x^2)$$

is (strictly) convex or concave on $\mathbb{R}^3$, and otherwise try to find the support restrictions on which one of the properties holds.

Exercise 5: Multivariate Differentiation

a.) Taylor Approximation

(i) Review: Univariate Taylor (online)

This exercise is meant as a means to get started with Taylor's theorem for those who did not cover it in their undergraduate or need to fresh up their memory. This less notation-intense context should help you deal with multivariate Taylor approximations as used in the next exercise. Feel free to skip ahead to *(ii)* if you are already well-familiar with Taylor approximations.

1.: First and Second Order Approximation. Compute the first and second order Taylor approximations to the exponential function $\exp : \mathbb{R} \mapsto \mathbb{R}_+, x \mapsto \exp(x)$ around $x_{0,1} = 1$ and $x_{0,2} = \ln(2)$. For $x_{0,1}$, illustrate the exponential function and its approximations. Is one globally preferable to the other, i.e. does it yield a weakly superior approximation everywhere?

2.: Higher Order Approximation. Compute the n-th order Taylor Approximation to the exponential function for $x_0 = 0$ for variable $n \in \mathbb{N}$. Can you find an infinite sum representation for the exponential function using polynomial terms?

(ii) Multivariate Taylor and Cobb-Douglas

For what follows, consider a household with Cobb-Douglas utility over consumption c and leisure l, i.e. $u(c,l) = c^{\alpha} l^{1-\alpha}$ with $\alpha \in (0,1)$. You can use that at points $(c,l) \neq (0,0)$, this function is infinitely many times differentiable.

1.: Approximation of Order 1. Compute the Taylor approximation of order 1 to $u(c,l)$ at $(c_0, l_0) = (1,1)$. For $\alpha = 1/2$, compare the approximated values for $(c,l) = (3/2, 1/2)$ and $(c,l) = (5, 4/5)$ to the true value of u.

2.: Approximation of Order 2. Compute the Taylor approximation of order 2 to $u(c,l)$ at $(c_0, l_0) = (1,1)$. For $\alpha = 1/2$, compare the approximated values for $(c,l) = (3/2, 1/2)$ and $(c,l) = (5, 4/5)$ to the true value of u.

Write down the first order Taylor expansion. You may use $\lambda \in (0,1)$ as an unknown variable.

b.) Matrix Functions (online)

Consider a matrix $A \in \mathbb{R}^{n \times n}$, $n \in \mathbb{N}$.

(i) Show that $\frac{d}{dx}(Ax) = A$.

(ii) What is the derivative of $f : \mathbb{R}^n \mapsto \mathbb{R}, x \mapsto x'Ax$?
Hint: Use *(i)* and the multivariate product rule.

(iii) If $A = \begin{pmatrix} 1 & \alpha \\ \beta & 4 \end{pmatrix}$, can you find values for α and β so that the second derivative of $x'Ax$ is positive definite everywhere? Can you find an alternative combination where A is positive semi-definite but not positive definite?