E600 Mathematics

Fall Semester 2020

In-class Exercises for Chapter 4

Discussed in class on Wednesday, September 23

Topics: Multivariate Differentiation and Integration

Problem 1: Solution Existence for Univariate Concave Functions

Consider a univariate, real-valued function $f : (\underline{x}, \bar{x}) \mapsto \mathbb{R}$, $\underline{x}, \bar{x} \in \mathbb{R}$ so that $\underline{x} < \bar{x}$. Suppose that
(i) f is once differentiable,
(ii) f is concave, and that
(iii) there exist $a, b, c \in (\underline{x}, \bar{x})$ with $a < b < c$ so that $f(a) < f(c) < f(b)$.
Can you argue that f assumes a global maximum on $(\underline{x}, \bar{x})$?

Hint 1. Recall that concavity is a desirable feature in unconstrained maximization.
Hint 2. Think about combining the Mean Value and Intermediate Value Theorem.

Problem 2: Existence of Solutions – Exploiting the Shape of the Function (online)

In practical applications, a common issue with the Weierstrass Extreme Value Theorem is that the support is not compact. For example, this is the case whenever we optimize over the whole $\mathbb{R}$ or $\mathbb{R}^n$ in unconstrained optimization or non-compact constraint sets, such as open intervals/balls. Fortunately, in many cases, we can *"compactify"* the domain and avoid issues with solution existence in an elegant way. In this exercise, you will establish a corollary of Weierstrass that is a concrete example of this method.

Corollary 1. ***(Optimizing a Univariate Function with Non-vanishing Limits)*** *Consider a function $f \in C^2(\mathbb{R})$, i.e. a function $f : \mathbb{R} \mapsto \mathbb{R}$ that is twice continuously differentiable. Assume that*

1. *There exist $a, b \in \mathbb{R}$ with $\lim_{x \to -\infty} f(x) = a$ and $\lim_{x \to \infty} f(x) = b$, that is, f does not diverge as $x \to \pm\infty$ but rather approaches fixed, real limits.*

2. *There exist* $c_1, c_2 > 0$ *such that either* $f'(x) > 0$ *for all* $x \in \mathbb{R}\backslash[-c_1, c_2]$ *(*Case 1*) or* $f'(x) < 0$ *for all* $x \in \mathbb{R}\backslash[-c_1, c_2]$ *(*Case 2*), that is, the sign of the derivative coincides for the limits* $x \to \pm\infty$*.*

3. *In* Case 1, $b \leq a$, *and in* Case 2, $a \leq b$.

Then, f *assumes both a global maximum and minimum, and the global extremizers are critical points of* f*.*

a.) Graphical Intuition

Assume that f has exactly two critical points (i.e. points with $f'(x) = 0$). Can you illustrate the intuition of this corollary graphically for *Case 1*?

b.) Type of Extrema

When f has exactly two critical points, can you say one is the global maximum/minimum of f depending on which case (*Case 1* or *Case 2*) you are in?

c.) Necessity of Limit Condition

Why do we need $b \leq a$ in *Case 1* and $a \leq b$ in *Case 2* to ensure existence of the global extrema?

d.) Formal Argument

Give a formal argument why the corollary holds, i.e. put the graphical intuition in a mathematical argument. You may restrict attention to *Case 1*.
Comment: An analogous argument can be made for *Case 2*. Because this case does not add an interesting particularity, we do not investigate the argument establishing it here.
Hint 1: Recall the definition of the limit $\lim_{x\to\infty} f(x)$: If $\lim_{x\to\infty} f(x) = c$, then

$$\forall \varepsilon > 0 \exists x^* \in \mathbb{R} : (\forall x > x^* : |f(x) - c| < \varepsilon)$$

Use this definition to restrict the investigation to a compact domain and apply Weierstrass.
Hint 2: It may be easier to investigate existence of the global maximum and minimum in isolation.

e.) Application

Solve

$$\max_{x\in\mathbb{R}} \frac{5x^2 - 2x}{6x^2 + 1} \quad \text{and} \quad \min_{x\in\mathbb{R}} \frac{5x^2 - 2x}{6x^2 + 1}$$

and, if there are global extremizers, compute the extreme values.

Problem 3: Saving Time in Optimization

Solve

$$\max \frac{4}{3}x^2 + y + xz \quad \text{s.t.} \quad \|(x,y,z)\|_2 \leq 1$$

where $\|\cdot\|_2$ is the Euclidean norm of the $\mathbb{R}^3$.

You will get a multitude solutions to the FOC. For any one of these, using the second order condition, you would have to check two determinants of the Bordered Hessian, one for a 3×3 and one for a 4×4 matrix. What may help to avoid this is that you can make an argument for solution existence, and also the Lagrangian multiplier condition.

Three simplifications and intuitions can be extremely helpful:

- If $\|(x,y,z)\|_2 < 1$, we can increase the objective varying marginally y.
- By norm non-negativity, $\|(x,y,z)\|_2 \leq 1$ is equivalent to $\|(x,y,z)\|_2^2 \leq 1^2$.
- Closed balls of the $\mathbb{R}^n$ are compact.