E600 Mathematics

Fall Semester 2020

In-class Exercises for Chapter 1

Discussed in class on Wednesday, September 16

Topics: Vector Spaces, Basis and Norms

Problem 1: Subspaces, Linear Dependence and Basis

a.) A Proper Subspace of $\mathbb{R}^3$

Prove that

$$S_2 := \{x = (x_1, x_2, x_3)' \in \mathbb{R}^3 : x_2 = 0\}$$

gives rise to a proper subspace of $\mathbb{R}^3$. What is its dimension?
Hint: use the linear combination definition of the subspace.

b.) Bases

Think of two different bases for $\mathbb{R}^3$. Include $b_1 = (1,1,0)'$ and $b_2 = (1,0,4)$ in the second.

Problem 2: Norm and Metric in Vector Spaces

a.) The Norms we use are actually Norms

Recall the most commonly used norms on $\mathbb{R}^2$:

- 1-norm ("Manhattan"): $\|x\|_1 = |x_1| + |x_2|$
- 2-norm ("Euclidean"): $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$
- infinity-norm ("Maximum"): $\|x\|_\infty = \max\{|x_1|, |x_2|\}$

(i) Show that the Euclidean norm constitutes a norm.
Hint: You may use the **Cauchy-Schwarz inequality** for the Euclidean space $(\mathbb{R}^n, \|\cdot\|_2)$, which states that for any $x, y \in \mathbb{R}^n$:

$$|x \cdot y| \leq \|x\|_2 \|y\|_2.$$

(ii) Except for the triangle inequality, the norm property proofs for the other norms are highly analogous. To convince yourself that the Maximum norm is also a norm, show that it satisfies the triangle inequality.

(iii) Sketch the unit-closed ball of these norms, i.e. $\bar{B}_1(0) = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$

Remark: The arguments establishing that the norms we considered here constitute norms on any $\mathbb{R}^n$, $n \in \mathbb{N}$ proceed in perfect analogy to the solutions of this exercise.

b.) Inverse Triangle Inequality

Let $(\mathbb{X}, \|\cdot\|)$ be a normed vector space. Show the inverse triangle inequality, that is, prove that

$$\forall x, y \in \mathbb{X} : \|x - y\| \geq \Big| \|x\| - \|y\| \Big|.$$

c.) Norm Continuity

Show that any norm is continuous. More formally: show that if $\mathbb{X} = (X, +, \cdot)$ is a vector space and $\|\cdot\|$ defines a norm on X, then it holds that for any $x_0 \in X$,

$$\forall \varepsilon > 0 \exists \delta > 0 : (x \in B_\delta(x_0) \Rightarrow \|x\| \in B_\varepsilon(\|x_0\|))$$

or equivalently

$$\forall \varepsilon > 0 \exists \delta > 0 : (\|x - y\| < \delta \Rightarrow \Big| \|x\| - \|y\| \Big| < \varepsilon).$$

Problem 3: Element-wise Applicability of the Multivariate Limit (online)

The course discussed the multivariate limit concept for general metric spaces. However, just from the definition, it is not too obvious how we should use it. Here, we turn to a rather powerful result, namely that the limit can be applied element-wise in metric spaces where the metric is p-norm induced.

Fact 1. In any normed space $(\mathbb{R}^n, \|\cdot\|_p)$, where $\|\cdot\|_p$ is a p-norm, $p \in \mathbb{N}$, $x^L \in \mathbb{R}^k$ is the limit of the sequence $\{x^n\}_{n \in \mathbb{N}}$ over $\mathbb{R}^k$ with $x^n = (x_1^n, \dots x_k^n)' \in \mathbb{R}^k$ for all $n \in \mathbb{N}$ if and only if all component sequences $\{x_j^n\}_{n \in \mathbb{N}}$, $j \in \{1, \dots, k\}$, are convergent, and $x^L = (\lim_{n \to \infty} x_1^n, \dots, \lim_{n \to \infty} x_k^n)$.

Prove Fact 1.

Hint 1: This is an equivalence proof, which typically means that you must show two directions separately.

Hint 2: On $\mathbb{R}^n$, all p-norms are equivalent in terms of convergence, continuity, etc.! Hence, it is sufficient to pick any one p-norm and show that Fact 1 holds for this specific case; convenient choices may be the ∞-norm or the 1-norm; the solution given here uses the ∞ norm.