

E600 Mathematics

Chapter 4: Optimization

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September 21, 2020

1. Introduction

Motivation

- Economics (broadly): study of optimal (“efficient”) allocations given limited resources → Sub-discipline of constrained optimization?!
 - Optimization: no better allocation ...
 - Constrained: taking as given constraints to the resources
- Examples
 - Utility maximization subject to budget constraint
 - Cost minimization given output target
 - Labor/leisure choice given time budget
 - (Tax-financed) Public good provision subject to government budget constraint
 - ...
- We focus on $f : X \mapsto \mathbb{R}$, $X \subseteq \mathbb{R}^n$ (why not $X \mapsto \mathbb{R}^m$?)

1. Introduction

Mathematical Constrained Optimization Problem

$$\begin{array}{ll} (\mathcal{P}_{min}) & \begin{array}{l} \text{minimize} \quad f(x) \\ x \in \text{dom}(f) \\ \text{subject to} \quad g_i(x) = 0, \quad i = 1, \dots, m. \\ \quad \quad \quad h_i(x) \leq 0, \quad i = 1, \dots, k. \end{array} \end{array}$$

$$\begin{array}{ll} (\mathcal{P}_{max}) & \begin{array}{l} \text{maximize} \quad f(x) \\ x \in \text{dom}(f) \\ \text{subject to} \quad g_i(x) = 0, \quad i = 1, \dots, m. \\ \quad \quad \quad h_i(x) \leq 0, \quad i = 1, \dots, k. \end{array} \end{array}$$

We focus on \mathcal{P}_{max} (easy to show equivalence of solutions)!

1. Introduction

Optimization: Roadmap

- Step 1: Unconstrained Problem – formal idea in simplest scenario
- Step 2: Problem with one equality constraint: justify Lagrangian formally
- Step 3: General equality-constrained problems: multivariate generalization of step 2
- Step 4: Take the intuition of 3 to solve general problems with inequality constraints
- Not covered: Convex Optimization
- Basically: know what you're doing when using the Lagrangian!

1. Introduction

Optimization: Concepts 1/2

- Maximum/minimum of a set X : (Extremum: min or max!)
 - $x = \max(X) \Leftrightarrow (x = \sup(X) \wedge x \in X)$ ($x \geq \tilde{x} \forall \tilde{x} \in X$)
 - $x = \min(X) \Leftrightarrow (x = \inf(X) \wedge x \in X)$ ($x \leq \tilde{x} \forall \tilde{x} \in X$)
- Local and global maximizers: $X \subseteq \mathbb{R}^n$, $f : X \mapsto \mathbb{R}$. Then, $x_0 \in X$ is a
 - Global maximizer for f if $\forall x \in X : f(x_0) \geq f(x)$
 - Local maximizer if $\exists \varepsilon > 0$ such that $\forall x \in X \cap B_\varepsilon(x_0) : f(x_0) \geq f(x)$
 - Strict versions: inequalities strict for all $x \neq x_0$
 - Global implies local
 - Graphically?

1. Introduction

Optimization: Concepts 2/2

- **Constraint set** of a problem \mathcal{P} : subset in domain of f

$$C(\mathcal{P}) := \{x \in X : ((\forall i \in \{1, \dots, m\} : g_i(x) = 0) \wedge (\forall j \in \{1, \dots, k\} : h_j(x) \leq 0))\}$$

- **Restricted function**: $A \subseteq \text{dom}(f)$

$$f|_A : A \mapsto \mathbb{R}, x \mapsto f(x)$$

- **Constrained** (strict) local/global **maximizer** of f in the problem \mathcal{P} : maximizer of $f|_{C(\mathcal{P})}$ (global maximizer = “**solution**”)
 - **arg max f** : Set of global maximizers of f ($\arg \max f \subseteq \text{dom}(f)$)
 - **max f** : Value at the maximum, $\max f = f(x)$, $x \in \arg \max f$
 - Solutions: $\arg \max f|_{C(\mathcal{P})}$, alternatively $\arg \max_{x \in C(\mathcal{P})} f(x)$

1. Introduction

Solutions and Problem Equivalence?

- Arg max and maximum summarized again:

$$\forall x^* \in \arg \max_{x \in C(\mathcal{P})} f(x) : (f(x^*) = \max_{x \in C(\mathcal{P})} f(x))$$

- If $\arg \max_{x \in C(\mathcal{P})} f(x) = \emptyset$, then $\max_{x \in C(\mathcal{P})} f(x)$ does not exist!
- If there is only a single arg max x^* , we write

$$x^* = \arg \max_{x \in C(\mathcal{P})} f(x)$$

- Constrained maximization problem: finding $\arg \max_{x \in C(\mathcal{P})} f(x)$!
- Actually: standard maximization of restricted function, but:
- restriction may not transfer appealing properties from f to $f|_{C(\mathcal{P})}$ (e.g. continuity) \rightarrow isolated investigation

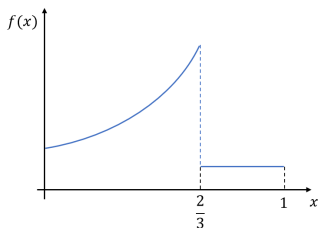
1. Introduction

Comment: Solution Existence

Theorem (Weierstrass Extreme Value Theorem)

Suppose that $X \subseteq \mathbb{R}^n$ is *compact*, and that $f : X \mapsto \mathbb{R}$ is *continuous*. Then, f assumes its maximum and minimum on X , such that $\arg \max_{x \in X} f(x) \neq \emptyset$ and $\arg \min_{x \in X} f(x) \neq \emptyset$.

- Why $\text{dom}(f)$ compact = closed + bounded? And why continuous?:



- Recall: intervals $[a, b] \subseteq \mathbb{R}$ are compact
- Example: $f : [0, 1] \mapsto \mathbb{R}$,

$$f(x) = \begin{cases} x^2 + 2 & x < 2/3 \\ 1 & x \geq 2/3 \end{cases}$$

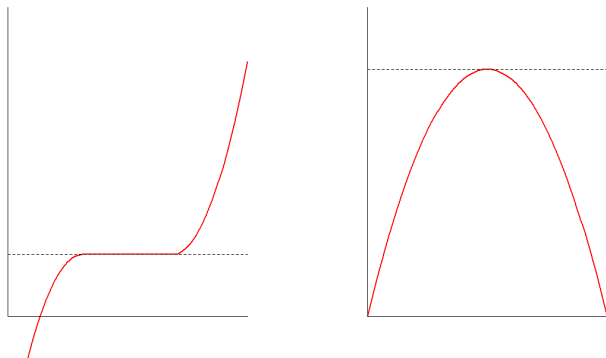
2. Unconstrained Optimization

Introduction

- General maximization approach
 - Find all local maximizers and see which are global (comparing values)
 - Ideally: **equivalent** conditions that can be easily checked
 - Alternatively: consider *larger* set of *potential* local maximizers (“candidates”) identified from **necessary** conditions
 - **sufficient** conditions/solution existence to identify maximum
- Equivalent conditions generally hard to come by (directly) → for now: necessary conditions (“If x_0 is a local maximizer, then...”)
- My focus: At least $f \in C^1(X)$, **points of discontinuity/non-differentiability are additional candidates generally!**
 - First: focus on interior points where derivative concept applies
 - Boundary points considered in isolation

2. Unconstrained Optimization

Necessary Conditions for Local Maximizers



- Local maximizer of continuous function: flat or hill \rightarrow zero slope!
- First order (first derivative) necessary condition: $f'(x^*) = 0$
- \mathbb{R}^n (“no slope in any direction”): $\nabla f(x^*) = \mathbf{0}$ (FOC)

2. Unconstrained Optimization

Necessary Conditions for Local Maximizers: FOC

Definition (Critical Point or Stationary Point)

Let $X \subseteq \mathbb{R}^n$, $f : X \mapsto \mathbb{R}$ and $x^* \in X$. Then, if f is differentiable at x^* and $\nabla f(x^*) = \mathbf{0}$, we call x^* a critical point of f or a stationary point of f .

- **Necessary FOC:** all *interior* local maxima are critical points
 - Proof: example for **contrapositive method**
- Not sufficient: more points feature $\nabla f(x) = 0$
 - Same logic applies to local minimum
 - “Saddle points”: minimum in one and maximum in other direction
- More insight from second derivative?
 - $f \in C^2(\mathbb{R})$: f' positive before and negative after local maximizer x^*
 $\Rightarrow f'$ decreasing around x^* : $f''(x^*) \leq 0$
 - Recall: definiteness \approx “sign” of symmetric matrix

2. Unconstrained Optimization

Necessary Conditions for Local Maximizers: SOC

- Second Order Necessary Condition (SOC): If $f \in C^2(X)$

(x^* is loc. maximizer) \Rightarrow ($H_f(x^*)$ is neg. semi-definite)

- For minimum: H_f pos. semi-definite
- Example: $f(x_1, x_2) = x_1^2 - x_2^2$
 - Gradient: $\nabla f(x_1, x_2) = (2x_1, -2x_2)$, Hessian:

$$H_f(x) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

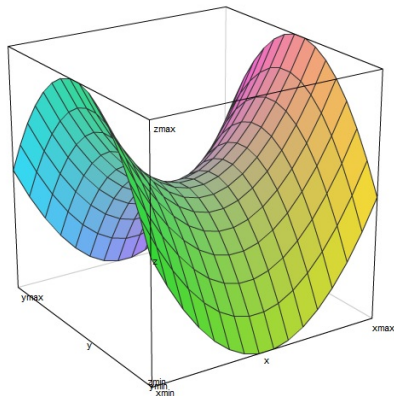
- Definiteness:

$$z' H_f(x) z = (z_1, z_2) \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (z_1, z_2) \begin{pmatrix} 2z_1 \\ -2z_2 \end{pmatrix} = 2(z_1^2 - z_2^2)$$

- Indefinite everywhere \Rightarrow no maximum or minimum!

2. Unconstrained Optimization

Necessary Conditions for Local Maximizers: SOC



- Left: graph of $f(x_1, x_2) = x_1^2 - x_2^2$
 - Indefinite Hessian: critical values may be *saddle points*
 - Why only necessary? Consider $n = 1$, $f(x) = x^3$
 - FOC:
 $f'(x^*) = 2(x^*)^2 = 0 \Leftrightarrow x^* = 0$
 - SOC: $f''(x^*) = 6x^*$, $f''(0) = 0$
 - $\forall v \in \mathbb{R} : v' f''(0) v = v^2 \cdot 0 = 0$
- SOC holds, but not local maximum!

2. Unconstrained Optimization

Sufficient Condition for Local Maximizers

Theorem (Unconstrained Local Maximum – Sufficient Condition)

Let $X \subseteq \mathbb{R}^n$, $f \in C^2(X)$ and $x^* \in \text{int}(X)$. Suppose that x^* is a critical point of f , and that $H_f(x^*)$ is negative definite. Then, x^* is a strict local maximizer of f .

- Minimum: FOC + Hessian *positive definiteness*
- Careful: what about x^* where
 - (i) the FOC holds: $\nabla f(x^*) = 0$,
 - (ii) $H_f(x^*)$ is negative semi-definite, but
 - (iii) $H_f(x^*)$ is not negative definite? \Rightarrow Cannot rule out as a solution, compare values to other candidates!
- Necess./suff. SOC proof: 1st order Taylor Expansion (see script)

2. Unconstrained Optimization

A Helpful Corollary

Corollary (Sufficiency for the Global Unconstrained Maximum)

Let $X \subseteq \mathbb{R}^n$ be a **convex** set, and $f \in C^2(X)$. Then, if f is concave and for $x^* \in \text{int}(X)$, it holds that $\nabla f(x^*) = \mathbf{0}$, then x^* is a **global** maximizer of f .

- Limit Behavior:
 - For univariate functions: $\lim_{x \rightarrow \pm\infty} f(x) = c$ breaks the largest interior maximum $f(x^*)$ as the global maximum if and only if $f(x^*) < c$
 - MV functions: usually consider asymptotically vanishing functions ($\lim_{\|x\| \rightarrow \infty} f(x) = -\infty$)
 - Otherwise (not needed often): compare largest interior maximum $f(x^*)$ to $\lim_{\lambda \rightarrow \infty} f(\lambda v^*(\lambda))$ where $v^*(\lambda)$ is the direction v that maximizes $f(\lambda v)$ for fixed v with $\|v\| = 1$
 - Example: online exercises

2. Unconstrained Optimization

Review: A Cookbook Recipe for the Global Maximum

- 1 Determine whether a solution exists at all (optional)
- 0 Collect border candidates: boundary points, non-diff'ability, limits
- 1 Interior solutions: Finding and eliminating candidates
 - If existence guaranteed and at any step, only one candidate (including border) remains, stop, you found the maximum!
 - Necessary FOC: Initial set of candidates = critical values
 - Necessary SOC: rule out those that violate it
 - If only one candidate (including border) remains
 - Existence guaranteed? Or: sufficient condition holds? Done ✓
- 2 Multiple candidates remaining: compare values, check existence if not already done

3. Constrained Optimization: One Equality Constraint

Introduction

- Last formal part, everything beyond will generalize rather easily
- We consider a problem of the form

$$(\mathcal{P}) \quad \underset{x \in C(\mathcal{P})}{\text{maximize}} \quad f(x) \quad \text{where} \quad C(\mathcal{P}) = \{x \in \text{dom}(f) : g(x) = 0\}$$

- Assume “smoothness regularity” for f and g , i.e. C^k if we need it

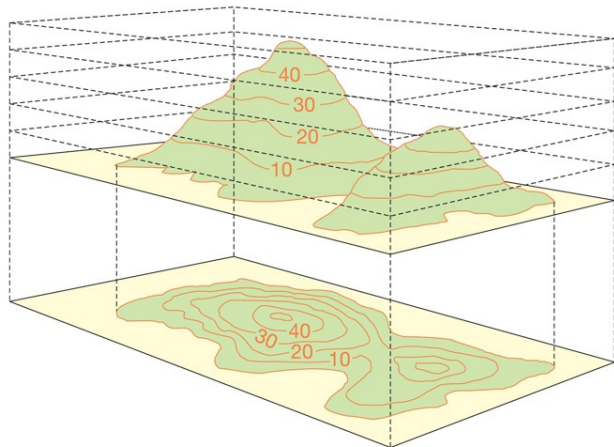
Definition (Level Set)

Let $X \subseteq \mathbb{R}^n$, $g : X \mapsto \mathbb{R}$, and $c \in \mathbb{R}$. Then, we call $L_c(g) = \{x \in X : g(x) = c\}$ the c -level set of g .

- Constraint set is zero-level set of g : $C(\mathcal{P}) = L_0(g)$
- Constrained maximization problem: find $\arg \max f|_{L_0(g)}$

3. Constrained Optimization: One Equality Constraint

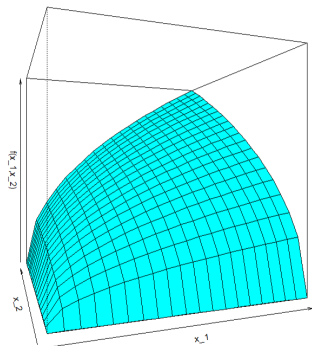
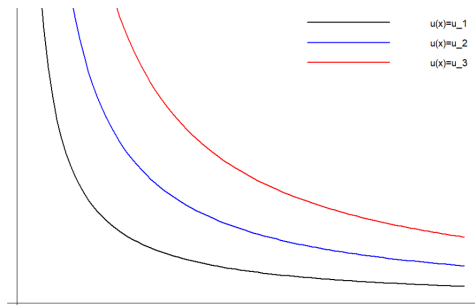
Level Sets 1/2



Level sets of “coordinates \mapsto height” (taken from http://canebrake13.com/fieldcraft/map_compass.php)

3. Constrained Optimization: One Equality Constraint

Level Sets 2/2



Indifference curves of utility function $u(x_1, x_2) = \sqrt{x_1 x_2}$

3. Constrained Optimization: One Equality Constraint

Intuition 1/2

- Recall: constrained local maximizer (on the level set):

$$x^* \in L_0(g) : (\exists \varepsilon > 0 : (\forall x \in B_\varepsilon(x^*) \cap L_0(g) : f(x^*) \geq f(x)))$$

- Deriving conditions like before: start from local maximizer x^* and consider neighborhoods $B_\varepsilon(x^*)$
- Only new issue: how to stay on the level set?
 - x^* lies on the level set: $g(x^*) = g(x_1^*, x_2^*, \dots, x_n^*) = 0$
 - Suppose we **marginally** vary $x_{-1}^* := (x_2^*, x_3^*, \dots, x_n^*)'$, but not x_1^*
 - If g is continuous, g should still be “really close” to zero
 - If $\frac{\partial g}{\partial x_1}(x^*) \neq 0$, we may pull g back to zero using x_1 :

$$g(h(x_2, x_3, \dots, x_n), x_2, x_3, \dots, x_n) = 0 \quad \text{for all } x_{-1} \in B_\varepsilon(x_{-1}^*)$$

3. Constrained Optimization: One Equality Constraint

Intuition 2/2

- h : **implicit function**

- Because $x^* = (h(x_{-1}^*), x_{-1}^*)$ is constrained local maximizer:

$$f(x^*) \geq f(h(x_{-1}), x_{-1}) \quad \text{for all } x_{-1} \in B_\varepsilon(x_{-1}^*)$$

- The procedure of course works with any j so that $\frac{\partial g}{\partial x_j}(x^*) \neq 0$
- Thus, x_{-1}^* is local maximizer of $f(h(x_{-1}), x_{-1})$
- Have reduced issue to **unconstrained problem that we can handle!**
- Label “implicit”: no explicit formula derived/known
- Complication: h unknown, does it have nice properties?

3. Constrained Optimization: One Equality Constraint

The Central Result

Theorem (Univariate Implicit Function Theorem)

Let $X_1 \subseteq \mathbb{R}$, $X_2 \subseteq \mathbb{R}^{n-1}$ and $X := X_1 \times X_2$, and $g : X \mapsto \mathbb{R}$. Suppose that $g \in C^1(X)$, and that for a $(y^*, z^*) \in X_1 \times X_2$, $g(y^*, z^*) = 0$. Then, if $\frac{\partial g}{\partial y}(y^*, z^*) \neq 0$, there exists an open set $U \subseteq \mathbb{R}^{n-1}$ such that $z^* \in U$ and $h : U \mapsto \mathbb{R}$ for which $y^* = h(z^*)$ and $\forall z \in U : g(h(z), z) = 0$. Moreover, it holds that $h \in C^1(U)$ with derivative

$$\nabla h(z) = - \left(\frac{\partial g}{\partial y}(h(z), z) \right)^{-1} \frac{\partial g}{\partial z}(h(z), z) \quad \forall z \in U.$$

- Super-technical, but nothing fancy:
 - $y \hat{=}$ x_1 : variable to be replaced, $z \hat{=}$ x_{-1} : remaining variables
 - Differentiability condition + “ g must move with y ”, yields continuous existence + formula derivative of h !
 - Note: $\frac{\partial g}{\partial z}$ is derivative w.r.t. $n - 1$ variables (gradient without $\frac{\partial g}{\partial y}$)

3. Constrained Optimization: One Equality Constraint

Putting Everything Together

- For a constrained local maximizer $x^* = (y^*, z^*)$, under the theorem's conditions,
 - $y^* = h(z^*)$ and z^* , and z^* is a local max. of $f(h(z), z)$
 - This gives the *necessary condition* $\frac{d}{dz} f(h(z), z) = \mathbf{0}$ for $z = z^*$
 - We can derive the **necessary FOC**

$$\exists \lambda \in \mathbb{R} : \nabla f(x^*) = \lambda \nabla g(x^*)$$

- Actually, we know that $\lambda = - \left(\frac{\partial g}{\partial y}(x^*) \right)^{-1} \frac{\partial g}{\partial z}(x^*)$, keep in mind!
- Additional border candidates (**singularities**): x^s so that $\nabla g(x^s) = 0$ (why?)

3. Constrained Optimization: One Equality Constraint

Summary FOC and Lagrangian

Theorem (Lagrange's Necessary First Order Condition)

Consider the constrained problem $\max_{x \in L_0(g)} f(x)$ where $X \subseteq \mathbb{R}^n$ and $f, g \in C^1(X)$. Let $x^* \in L_0(g)$ and suppose that $\nabla g(x^*) \neq \mathbf{0}$. Then, x^* is a local maximizer of the constrained problem only if there exists $\lambda \in \mathbb{R} : \nabla f(x^*) = \lambda \nabla g(x^*)$. If such $\lambda \in \mathbb{R}$ exists, we call it the *Lagrange multiplier* associated with x^* .

- Equivalently: $\exists \lambda \in \mathbb{R} : \nabla f(x^*) - \lambda \nabla g(x^*) = \mathbf{0}$
- Lagrangian function (or: "Lagrangian")

$$\mathcal{L}(\lambda, x) = f(x) - \lambda g(x)$$

- The FOC for λ is $g(x) = 0$, i.e. $x \in L_0(g)$
- Thus, $x^* \in X$ satisfies the necessary FOC if and only if for a $\lambda \in \mathbb{R}$, (λ, x) is a critical value of the Lagrangian function!

3. Constrained Optimization: One Equality Constraint

The Lagrangian: Second Order Conditions

- Constrained maximization = Lagrangian maximization?
 - Identical first order necessary condition
 - But: Lagrangian has only saddle points (see script)
 - Can still derive sufficiency criterion for constrained maxima/minima from second derivative = Hessian of the Lagrangian
 - Here: no necessary SOC!
 - Issue: rather ugly conditions
 - ... but we might also use our intuition for the problem
 - One last concept...

3. Constrained Optimization: One Equality Constraint

Definition (Leading Principal Minor)

Consider a symmetric matrix $A = (a_{ij})_{i,j \in \{1, \dots, n\}} \in \mathbb{R}^{n \times n}$. Then, for $k \leq n$, the k -th leading principal minor of A , or the leading principal minor of A of order k is the matrix obtained from eliminating all rows and columns with index above k from A , i.e. the matrix $M_k^A = (a_{ij})_{i,j \in \{1, \dots, k\}} \in \mathbb{R}^{k \times k}$.

Example:

$$A = \begin{pmatrix} 1 & 4 & 3 & 2 \\ 2 & 0 & 0 & 0 \\ 3 & 4 & -1 & -2 \\ 0 & 1 & e & \pi \end{pmatrix}.$$

Then, the leading principal minors of order 1, 2, 3 and 4, respectively, are

$$M_1^A = (1), \quad M_2^A = \begin{pmatrix} 1 & 4 \\ 2 & 0 \end{pmatrix}, \quad M_3^A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 0 & 0 \\ 3 & 4 & -1 \end{pmatrix}, \quad M_4^A = A.$$

3. Constrained Optimization: One Equality Constraint

Sufficiency in the Lagrangian Problem

Theorem (Lagrange's Sufficient Conditions)

Consider the constrained problem $\max_{x \in L_0(g)} f(x)$ where $X \subseteq \mathbb{R}^n$ and $f, g \in C^2(X)$. Let $x^* \in L_0(g)$ and $\lambda^* \in \mathbb{R}$ such that $\nabla f(x^*) = \lambda^* \nabla g(x^*)$ and $g(x^*) = 0$. If $m = 1$ is the number of equality constraints, denote by $M_{n-m+1}^{H_{\mathcal{L}}}(\lambda^*, x^*), \dots, M_n^{H_{\mathcal{L}}}(\lambda^*, x^*)$ the last $n - m$ principal minors of $H_{\mathcal{L}}(\lambda^*, x^*)$. If

- $\forall j \in \{n - m + 1, \dots, n\} : \text{sgn}(\det(M_j^{H_{\mathcal{L}}})) = (-1)^m$, then x is a local minimizer of the constrained problem.
- $\forall j \in \{n - m + 1, \dots, n\} : \text{sgn}(\det(M_j^{H_{\mathcal{L}}})) = (-1)^j$, then x is a local maximizer of the constrained problem.

$$\text{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

3. Constrained Optimization: One Equality Constraint

Second Order Condition: Comments

- $H_{\mathcal{L}}$: **Bordered Hessian**, structure:

$$\begin{aligned} H_{\mathcal{L}}(\lambda, x) &= \begin{pmatrix} \frac{\partial^2 \mathcal{L}}{\partial \lambda^2}(\lambda, x) & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x}(\lambda, x) \\ \frac{\partial^2 \mathcal{L}}{\partial x \partial \lambda}(\lambda, x) & \frac{\partial^2 \mathcal{L}}{\partial x^2}(\lambda, x) \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\nabla g(x) \\ -(\nabla g(x))' & H_f(x) - \lambda H_g(x) \end{pmatrix} \end{aligned}$$

- Can use sign of submatrices to determine extremum type
- No necessary SOC: typically, not too many candidates anyways

3. Constrained Optimization: One Equality Constraint

Lagrangian: An Example

Find the vector with minimum Euclidean length $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$ in the \mathbb{R}^2 that satisfies $x_1 + x_2 = 1$, i.e. solve

$$\max_{x \in \mathbb{R}^2} -\|x\|_2 \quad \text{subject to} \quad x_1 + x_2 = 1$$

or equivalently,

$$\max_{x \in \mathbb{R}^2} -(x_1^2 + x_2^2) \quad \text{subject to} \quad x_1 + x_2 - 1 = 0$$

3. Constrained Optimization: One Equality Constraint

Lagrangian: Intuition

- Why $n - m$ last principal minors?
 - Generally: first derivative condition $\text{rk } J_g(x^*) = m$ ($m = 1$?)
 - constraints *linearly independent* at x^*
 - m constraints restrict x in m dimensions, $n - m$ “free variables”
 - One condition for every free direction like before! ($\nabla f / H_f$)
- Lagrangian Multiplier: value cost of the constraint
 - Example: shadow cost of the budget constraint
 - Here: $\lambda =$ change in objective due to marginally “relaxing” constraint
 - $\lambda < 0 \rightarrow$ constraint limits our ability to *lower* objective; not maximizer!

3. Constrained Optimization: One Equality Constraint

Lagrangian Multipliers as a Necessary Condition

- (When) Does the multiplier trick really work?
 - Need equivalence to inequality-constrained problem ($g(x) \leq 0$!)
 - Value cost interpretation requires differentiability of $x^*(c)$
- We can show:
 - $\lambda^* < 0$ rules out local maximizers
 - $\lambda^* > 0$ rules out local minimizers
- Interpretation: “relaxing” the constraint (Δc)?
 - $g(x) = \tilde{g}(x) - c \leq 0 \rightarrow \Delta c > 0$
 - $g(x) = c - \tilde{g}(x) \leq 0 \rightarrow \Delta c < 0$
 - Second variant “inverts” value cost logic; $\lambda =$ effect of increasing $-c$!
- Application shown in exercise sessions

3. Constrained Optimization: One Equality Constraint

Lagrangian: Generalization – Idea

- Multiple equality constraints $m \in \mathbb{N}$: stacked in

$$g = \begin{pmatrix} g^1 \\ \vdots \\ g^m \end{pmatrix}$$

so that the constraint becomes $g(x) = \mathbf{0}$

- Looks a lot like single constraint (recall: vector space)
 - *Only* adjustment: varying x_i generally moves all m directions of $g(x)$
- need to adjust m arguments in the implicit function:

$$g(h(x_{-m}), x_{-m}) = 0 \quad \text{for } x_{-m} = (x_{n-m+1}, x_{n-m+2}, \dots, x_n)$$

- Adjustment must reach all m directions in $\text{im}(g)$: $\text{rk}(J_g(x^*)) = m$
- Inequality constraints $h(x) \leq 0$: if they don't hold with equality ("bind") at x^* , they are irrelevant → multiplier μ satisfies $\mu h(x) = 0$

4. Constrained Optimization: Multiple Equality Constraints

Theorem (Multivariate Implicit Function Theorem)

Let $X_1 \subseteq \mathbb{R}^m$, $X_2 \subseteq \mathbb{R}^{n-m}$ and $X := X_1 \times X_2$, and $g : X \mapsto \mathbb{R}^m$. Suppose that $g \in C^1(X, \mathbb{R}^m)$, and that for a $(y^*, z^*) \in X_1 \times X_2$, $g(y^*, z^*) = \mathbf{0}$.

Then, if $\text{rk}\left(\frac{\partial g}{\partial y}(y^*, z^*)\right) = m$, there exists an open set $U \subseteq \mathbb{R}^{n-1}$ such that $z^* \in U$ and $h : U \mapsto \mathbb{R}^m$ for which $y^* = h(z^*)$ and $\forall z \in U : g(h(z), z) = \mathbf{0}$. Moreover, it holds that $h \in C^1(U, \mathbb{R}^m)$ with derivative

$$J_h(z) = - \left(\frac{\partial g}{\partial y}(h(z), z) \right)^{-1} \frac{\partial g}{\partial z}(h(z), z) \quad \forall z \in U.$$

- Everything in vectors, the derivative of h is now a Jacobian
- Row rank condition for the “partial Jacobian” can be satisfied if $\text{rk } J_g(x^*) = m$ (row rank = column rank)

4. Constrained Optimization: Multiple Equality Constraints

Theorem (Lagrange's Multiple First Order Necessary Condition)

Consider the constrained problem $\max_{x \in L_0(g)} f(x)$ where $X \subseteq \mathbb{R}^n$ and $f \in C^1(X)$, $g \in C^1(X, \mathbb{R}^m)$. Let $x^* \in L_0(g)$ and suppose that $\text{rk}(J_g(x^*)) = m$. Then, x^* is a local maximizer of the constrained problem only if there exists $\Lambda = (\lambda_1, \dots, \lambda_m)' \in \mathbb{R}^m$: $\nabla f(x^*) = \Lambda' J_g(x^*)$. If such $\Lambda \in \mathbb{R}$ exists, we call λ_i the Lagrange multiplier associated with x^* for the i -th constraint.

Theorem (Lagrange's Multiple Sufficient Conditions)

Suppose additionally that $f \in C^2(X)$, $g \in C^2(X, \mathbb{R}^m)$, and that (Λ^*, x^*) is a critical point of the Lagrangian function, i.e. $\nabla f(x^*) = (\Lambda^*)' J_g(x^*)$ and $g(x^*) = \mathbf{0}$. Denote by $M_{n-m+1}^{H_{\mathcal{L}}}(\lambda^*, x^*), \dots, M_n^{H_{\mathcal{L}}}(\lambda^*, x^*)$ the last $n - m$ principal minors of $H_{\mathcal{L}}(\lambda^*, x^*)$. If

- $\forall j \in \{n - m + 1, \dots, n\} : \text{sgn}(\det(M_j^{H_{\mathcal{L}}})) = (-1)^m$, then x is a local minimizer of the constrained problem.
- $\forall j \in \{n - m + 1, \dots, n\} : \text{sgn}(\det(M_j^{H_{\mathcal{L}}})) = (-1)^j$, then x is a local maximizer of the constrained problem.

4. Constrained Optimization: Multiple Equality Constraints

A Remark on the FOC

- FOC for x^* : $\nabla f(x^*) = \Lambda' J_g(x^*)$ may look a bit weird
- $\Lambda = (\lambda_1, \dots, \lambda_m)$, multiplying out gives

$$\nabla f(x^*) = \sum_{i=1}^m \lambda_i \nabla g_i(x^*)$$

- More obvious generalization of univariate case

4. Constrained Optimization: Multiple Equality Constraints

Review: A Cookbook Recipe for the Constrained Global Maximum

- ① Determine whether a solution exists at all (optional)
- ② Collect border candidates: boundary points, non-diff'ability, **singularities of the level set**
- ③ Interior solutions: Finding and eliminating candidates
 - Necessary FOC: Initial set of candidates = **Lagrangian** critical values (No necessary SOC!)
 - Single candidate (including border) + existence guaranteed? **Done ✓**
 - Sufficient FOC: rule out those identified as strict local minimum
Alternatively: use Lagrangian multipliers?
 - If only one candidate (including border) remains
 - Existence guaranteed? Or: sufficient condition holds? **Done ✓**
- ④ Multiple candidates remaining: compare values, check existence if not already done; limits when constraint set is not bounded

5. Constrained Optimization: Inequality Constraints

Introduction

- Recall: maximization problem \mathcal{P} with general constraint set

$$C(\mathcal{P}) := \{x \in X : ((\forall i \in \{1, \dots, m\} : g_i(x) = 0) \wedge (\forall j \in \{1, \dots, k\} : h_j(x) \leq 0))\}$$

- Examples

- (Government) Budget constraints: $p \cdot x \leq y$
- Non-negativity constraints: $\forall i \in \{1, \dots, n\} : x_i \geq 0$
- Production possibility frontier
- etc.

- Our approach:

- 1 Formal theorem to deal with constraints, a bit messy
- 2 Replace inequality with equality or remove the constraint altogether

5. Constrained Optimization: Inequality Constraints

Theorem for Inequality Constraints: Intuition

- Recall Lagrangian FOC: (“FOC + feasibility”)

$$\nabla f(x^*) = \sum_{i=1}^m \lambda_i \nabla g_i(x^*) \quad \wedge \quad \forall i \in \{1, \dots, n\} : g_i(x) = 0$$

- Binding constraint: $h_j(x^*) = 0$ “like an equality constraint” at x^*
- Slack constraint $h_j(x^*) < 0$ has no value cost: zero multiplier μ_j
 - “Complementary slackness” condition: $\mu_j h_j(x^*) = 0$
 - Set of binding inequality constraints varies across candidates, but...
 - $\mu_j h_j(x^*)$ does not! Lagrangian FOC is always

$$\nabla f(x^*) = \sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{j=1}^k \mu_j h_j(x) \quad \wedge \quad \forall i \in \{1, \dots, n\} : g_i(x) = 0$$

because $\mu_j = 0$ for all non-binding (irrelevant) constraints

- Add feasibility: $h_j(x^*) \leq 0$ for non-binding constraints and we're done

5. Constrained Optimization: Inequality Constraints

Theorem (Karush-Kuhn-Tucker Theorem)

For $\Lambda = (\lambda_1, \dots, \lambda_m)' \in \mathbb{R}^m$ and $\mu = (\mu_1, \dots, \mu_k)' \in \mathbb{R}^k$, consider the **optimality conditions**

- (i) (Feasibility) $\forall j \in \{1, \dots, k\} : h_j(x) \leq 0$ and $\forall i \in \{1, \dots, m\} : g_i(x) = 0$,
- (ii) (FOC for x) $\nabla f(x) = \sum_{i=1}^m \lambda_i \nabla g_i(x) + \sum_{j=1}^k \mu_j \nabla h_j(x)$,
- (iii) (Complementary Slackness) $\forall j \in \{1, \dots, k\} : \mu_j h_j(x) = 0$.

Then, if $x^* \in \text{dom}(f)$ is a local maximum of the constrained problem for which **the set $\{\nabla h_j(x^*) : h_j(x^*) = 0\} \cup \{\nabla g_i(x^*) : i \in \{1, \dots, m\}\}$ is linearly independent**, there exist $\Lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^k$ such that (x^*, Λ^*, μ^*) satisfy the optimality conditions.

- In words: the optimality conditions are (first-order) **necessary**
- **Rank condition**: set of binding constraints varies across locations

5. Constrained Optimization: Inequality Constraints

From Kuhn-Tucker to Simpler Methods

- Necessary FOC may be sufficient if solution exists and a unique candidate remains
- Sufficient conditions: convex optimization (concave objective, quasi-convex inequality constraints, “no” equality constraints)
 - More details in script
- We may be able to simplify the issue to an equality-constrained one!
- Intuition: set of solutions unchanged by imposing equality or removing constraint

5. Constrained Optimization: Inequality Constraints

From Kuhn-Tucker to Simpler Methods

- Imposing equality: “not really an inequality constraint at all”
 - Formally: x^* local maximizer \Rightarrow constraint binding
 - Contrapositive: constraint not binding $\Rightarrow x^*$ not local maximizer
 - Show either and you may impose equality
- Dropping the constraint: “irrelevant”
 - Formally: constraint binding $\Rightarrow x^*$ not local maximizer

\Rightarrow Problem modification preserves the set of solutions!

- Example: constrained utility maximization with “regular” utility function ($\frac{\partial u}{\partial x_j}$ strictly monotonically increasing, $\lim_{x_j \rightarrow 0} \frac{\partial u}{\partial x_j}(x) = \infty$)

5. Constrained Optimization: Inequality Constraints

That's it for the lectures!