E600 Mathematics

Welcome and Chapter 0: Fundamentals of Mathematics

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Welcome to E600 Mathematics!

Overview

- What is E600 Mathematics?
  - Introductory MSc module for mathematical background
  - Covers key math. areas that the MSc economics in Mannheim (and most other programs) draw from
  - Course is not mandatory, **no exam**
    - $\Rightarrow$ All your efforts are a voluntary (but rewarding) investment into the next two years!
  - more detail in a minute...
- Instructor: Martin Reinhard
  - 4th year PhD student at CDSE
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  - no scheduled office hours, individual meetings can be arranged

1. Logistics and Organization

What and Why?

- Main Course Objectives
  - Review central mathematical concepts for economists
  - Get more familiar and comfortable with logic and notation
  - Know and be able to apply main theorems
  - Have a rough idea of why they are true (not main focus)
- Why care?
  - Mannheim has a strong quantitative orientation
  - Logical reasoning and mathematical methods are central to economics

$\Rightarrow$ No "self-motivated" course, rather help you get started in the MSc program

1. Logistics and Organization

When?

Time	Mon, Sep 14	Wed, Sep 16	Fri, Sep 18	Mon, Sep 21	Wed, Sep 23
9:00 - 10:30	Welcome & Course Logistics	Lecture Ch. 1	Lecture Ch. 2	Lecture Ch. 3	Lecture Ch. 4
	Review of Ch. 0	Solutions Self-Study 1	Solutions Self-Study 2	Solutions Self-Study 3	Solutions Self-Study 4
10:45 - 12:15	In-class Exercises Ch. 0				
	Lecture Ch. 1	In-class Exercises Ch. 1	In-class Exercises Ch. 2	In-class Exercises Ch. 3	In-class Exercises Ch. 4
13:15 - 14:45		Lecture Ch. 2	Lecture Ch. 3	Lecture Ch. 4	International Office Event for International Students
15:00 - 16:30					No class/exercise

- ≈ 4 blocks per chapter
- 60% lecture, 40% exercise
- Days without class: self-study exercises

- Schedule is broad outline, may take more/less time according to demand
- Wed, Sep. 23: Q&A session after In-class 4

1. Logistics and Organization

How? 1/2

- Web: `e600.uni-mannheim.de` (preliminary: `e600math.epizy.com`)
  - Full text for all chapters as Online Course
    - Interactive design: short questions, review quizzes
    - Focused on core competencies: few proofs, involved concepts left out
  - Additional practice problems for each chapter
- Course Material (available at `e600.uni-mannheim.de/downloads`)
  - Self-study and in-class exercise sheets, slides
  - Companion script: everything we do in class (and more) + all proofs
- Online Course & Script
  - *comprehensive* resources: meant as your MSc companion to consult for any issue related to mathematics
  - *self-sufficient* resources: consistent notation, rely only on known/previously introduced concepts

1. Logistics and Organization

How? 2/2

- Self-study
  - Online course + script, online exercises
  - Work through chapters/topics selectively & at your own pace
- Attend class
  - Class-based module “forces” you to spend time with the material
  - I can help you to read between the lines
  - Don’t hesitate to **ask questions!** (also after class)
  - Learn how to come up with solutions approaches in the exercise sessions
- Work on self-study (“homework”) problems

1. Logistics and Organization

Self-study Exercises

- Days without class: review and work on self-study exercises
  - Expectation: understand all problems, solve the easier ones
  - No need to hand in anything
  - Solutions discussed in class on the next day
- Work in groups
  - Get to know your fellow students (better)
  - Divide problems across group members
  - Do you want/need a group?
- Please also take the time to fill in a short feedback survey

1. Logistics and Organization

Expectation

- E600 has **a lot of material**
  - many important concepts to cover in little time
  - not all is new; you need not know everything by the end of this class
- After this class, you should...
  - not be afraid of mathematical notation and formal arguments
  - have heard of the math. concepts central to economics and know where to look them up
  - everything else is a plus!
- In times of Corona: please interact!
  - ask questions during and after class
  - work in groups
  - give feedback through the website

2. Fundamentals: Why Math?

aka "Why not a review of Micro/Macro/Econometrics?"

- Economists (among many other professions) put great emphasis on logical, rational reasoning – math is the purest form of this
- Economic investigations: mathematical or statistical models
- Math is actually less messy than the real world
  - No conceptual ambiguity: "fair economic system" vs. "Derivative of a given function f"? (existence?)
  - Full information: statements can be proved/disproved with certainty

$\Rightarrow$ "lab conditions" to test logical reasoning

- Math = "language" very similar to normal English
  - Vocabulary: words + notation; sentence = "statement"
  - Most important "sentences": quantifiers
    - Existential quantifier $\exists \ldots$
    - Universal quantifier $\forall \ldots$
    - Example: $\forall n \in \mathbb{N} : n \geq 0$
  - Grammar: rules that determine whether statements are meaningful
- Logical argument: statement about relationship of statements
  - English: "Any animal that is not a cat and a dog is a horse."
  - Math: $\forall a \in A : ((a \notin C \wedge a \notin D) \Rightarrow a \in H)$

3. Fundamentals: Language and Logic

Vocab and Grammar of Mathematics 2/3

- Particularities of the mathematical language
  - The logical “or” ($\vee$) reads as “and/or” so that e.g. $(x > 4 \vee x > 2)$ is true both for $x = 3$ and $x = 5$
  - “greater than” includes equality, $>$ reads as “strictly greater than”
- Statements: meaningful (grammar) and true (content)?

Bananas blue	$\forall x \in \mathbb{R} : x > y$
Bananas are blue	$\forall x \in \mathbb{R} : (\forall y \in \mathbb{R} : x > y)$
Bananas are yellow	$\forall x \in \mathbb{R} : (\exists y \in \mathbb{R} : x > y)$

- Negation: usually more direct when re-written
  - E.g.: $\neg(x \in A) \Leftrightarrow x \notin A$
  - Quantifiers: $\neg(\forall \ldots : S) \Leftrightarrow (\exists \ldots : \neg S)$ and vice versa
    - E.g.: $\neg(\exists n \in \mathbb{N} : n < 0) \Leftrightarrow (\forall n \in \mathbb{N} : n \geq 0)$

4. Fundamentals: Arguments and Reasoning

Vocab and Grammar of Mathematics 3/3

- Classic argument: 2 premises imply one conclusion
  - (P1: All teachers like coffee, P2: I am a teacher $\Rightarrow$ C: I like coffee)
- More complex arguments: multiple premises and/or conclusions; assertion of *equivalence* rather than implication
- Argument: valid and sound?
  - valid: asserted relationship is correct
  - sound: valid argument and the premises are true
  - Ex. 1: (Mannheim is a German city $\wedge$ Germany is an Asian country) $\Rightarrow$ Mannheim is an Asian city
  - Ex. 2: (Mannheim is a German city $\wedge$ Germany is a European country) $\Rightarrow$ Mannheim is a European city
- Mathematical proof: establish non-obvious validity of argument
- Caution: arguments do not address causation (e.g. coffee)!

- Consider a statement/"event" E
  - Sufficient conditions S *imply* E: S guarantees E ("if")
  - Necessary conditions N *are implied by* E: $\neg N$ rules out E ("only if")
  - Necessary *and* sufficient = equivalent ("if and only if")
- Example: $E := (x \in \mathbb{N})$
  - sufficient: $x = 2$
  - necessary: $x \in \mathbb{Z}$
  - equivalent: $(x \in \mathbb{Z} \wedge x \geq 0)$

5. Fundamentals: Sets

- Set = collection of distinct objects ("elements")
  - $\{1, 2, \pi\}$ vs. $\{1, 1, \pi\}$ vs. $\{n \in \mathbb{N} : n > 10\}$
  - Elements can be sets, matrices, functions, whatever
  - Key concept: subset $A \subseteq S \Leftrightarrow (x \in A \Rightarrow x \in S)$; proper subset $A \subset S$
- Intervals: $I \subseteq \mathbb{R}$, write $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
  - (right-/left-) closed and open interval: $(a, b]$, $[a, b)$, $[a, b]$
- Set relations: equal vs. disjoint, complement
- Operations
  - Union $A \cup B$, Intersection $A \cap B$, Set difference $A \backslash B$ (circles)
  - Power set $\mathcal{P}(A) = \{S \subseteq X : S \subseteq A\}$: set of subsets
  - Index sets for *sets/sequences of sets*: $\bigcup_{i \in I} A_i$, $\bigcap_{i \in I} A_i$ (e.g. $I = \mathbb{N}$; what does pairwise disjoint mean?)

- Functions define a relation between sets
  - Usual notation: $f : X \mapsto Y, x \mapsto y = f(x)$
    - X: domain, Y: codomain
    - Mapping rule "$x \mapsto y = f(x)$" relates each $x \in X$ to exactly one $y \in Y$
  - Relation = collection of pairs $(x, y) = (x, f(x))$ in $X \times Y$:

$$G(f) = \{(x, y) \in X \times Y : y = f(x)\} = \{(x, f(x)) : x \in X\}$$

  - The *set* $G(f)$ is called the graph of f
- Concepts:
  - f is a function, $f(x)$ is not! (depending on context: element of codomain or mapping rule)
  - Relations are sets

- Consider $f : X \mapsto Y$ (e.g. $f : \mathbb{R}_+ \mapsto \mathbb{R}, x \mapsto \sqrt{x}$)
    - Image of $A \subseteq X$, preimage of $B \subseteq Y$ ($A = [1, 4]$, $B = [4, 16)$)
    - $g : Y \mapsto Z$; composition $g \circ f$
    - Inverse function f^{-1} of f: $f^{-1} \circ f = f \circ f^{-1} = Id$
        - Id: identity function for which $Id(x) = x$
        - Existence depends on *bijectivity*
- The preimage and the inverse function are **fundamentally different objects**!

7. Fundamentals: Limits and Continuity in $\mathbb{R}$

- Sequence: $\lim_{n\to\infty} x_n = x \Leftrightarrow |x_n - x| \overset{n\to\infty}{\to} 0$
- Function $f : X \mapsto \mathbb{R}$ where $X \subseteq \mathbb{R}$:

$$\lim_{x\to a} f(x) = f_a \in \mathbb{R} \quad \Leftrightarrow \quad |f(x) - f_a| \underset{x\neq a}{\overset{x\to a}{\to}} 0$$

- Notational conventions
  - Divergence $\lim_{x\to a} f(x) = \pm\infty$ (e.g. $\lim_{x\to 0} \frac{1}{|x|} = \infty$)
  - Asymptotic behavior: $\lim_{x\to\infty} f(x) = c$: $|f(x) - c| < \varepsilon$ for a $x > C(\varepsilon)$
- Continuity of f at $a \in X$: $\lim_{x\to a} f(x) = f(a)$
  - Examples for discontinuity: $f(x) = \mathbb{1}[x > 0]$, $f(x) = \mathbb{1}[x = 0]$ (draw!)
  - Characterization: left and right limit exist at a and are equal to $f(a)$
  - Prove continuity: formal definition, left/right limits
  - Disprove: left $\neq$ right limit, sequence $a_n \to a$ where $f(a_n) \not\to f(a)$

- lim . . . can be pulled into any continuous expression (including sums, products)
- Differentiable Functions: L'Hôpital's rule
  - e.g. $\lim_{x \to 0} \frac{\sin(x)}{x}$
- Sequences: Sandwich Theorem. Suppose $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \bar{x}$ and

$$\forall n \geq N \in \mathbb{N} : y_n \leq x_n \leq z_n.$$

  Then, $\lim_{n\to\infty} x_n = \bar{x}$.
  - e.g. $\lim_{n\to\infty} -\frac{1}{n^2+4n+25}$

8. Fundamentals: Mathematical Proof

Our Approach

- Recall: proofs establish non-obvious validity of mathematical statement/fact
  - understanding them may improve your understanding of the fact
  - some are tedious (relative to the importance of the fact)
- How do we proceed?
  - The companion script gives all proofs in consistent notation
  - in class: easier proofs for the most central results
  - also: examples of methods of proof so that you have seen them once
    - Proof by induction
    - Equivalence proof in two steps
    - Contrapositive method of proof
    - ...