E600 MATHEMATICS

Problem Set 4: Unconstrained and Constrained Optimization

Fall Semester 2019, course taught by: Martin Reinhard

Problem 1: Writing Down a Problem and Solution Existence

a.) The Economic Problem of a Lazy Person

Consider an economic agent who, for some reason, derives utility from time not spent working, and from consumption. Formalize the problem the agent faces when

- the maximum number of hours available is H,
- an hour of work gives wage $w > 0$,
- one unit of the (single) consumption good costs price $p > 0$.

Supposing that we deal with a standard economic utility function, i.e. that marginal utility of both consumption and leisure is strictly increasing with left limit ∞, can you formulate an equivalent unconstrained problem?

b.) Weierstrass Extreme Value Theorem

Does the Weierstrass Extreme Value Theorem apply to the functions

1. $f : \mathbb{R} \mapsto \mathbb{R},\ x \mapsto x^3$
2. $f : (0, \pi) \mapsto \mathbb{R},\ x \mapsto \cos(x)$
3. $f : \{x \in \mathbb{R}^2 : (1,2) \cdot x \leq 5\} \mapsto \mathbb{R},\ x \mapsto x_1 + x_2$
4. $f : [-1,1] \mapsto \mathbb{R},\ x \mapsto \mathbb{1}[x > 0]$
5. $f : [0, \pi] \mapsto \mathbb{R},\ x \mapsto (\cos(x) + 2)^{\sin(x)}$
6. $f : \bar{B}_1(\mathbf{0}) \mapsto \mathbb{R},\ x \mapsto x'x$ where $\mathbf{0} \in \mathbb{R}^5$

Is there an example in the functions 1.-6. that demonstrates that the Weierstrass Extreme Value Theorem only formulates only a sufficient, but not an equivalent condition for existence of the global extreme values?

c.) Polynomial Functions and Extreme Values

Consider a polynomial of order $N \in \mathbb{N}$, $f(x) = \sum_{n=0}^{N} \lambda_k x^n$ with coefficients $\{\lambda_k\}_{n=0}^{N} \subseteq \mathbb{R}^{N+1}$. When does the restriction of this function to $[-4, 2]$ have a maximum?

Problem 2: Solution Existence for Univariate Concave Functions

Consider a univariate, real-valued function $f : (\underline{x}, \bar{x}) \mapsto \mathbb{R}$, $\underline{x}, \bar{x} \in \mathbb{R}$ so that $\underline{x} < \bar{x}$. Suppose that
(i) f is once differentiable,
(ii) f is concave, and that
(iii) there exist $a, b, c \in (\underline{x}, \bar{x})$ with $a < b < c$ so that $f(a) < f(c) < f(b)$.
Can you argue that f assumes a global maximum on $(\underline{x}, \bar{x})$?

Hint 1. Recall that concavity is a really desirable feature in unconstrained maximization problems.

Hint 2. The "intermediate value theorem", which we can think of as an analogy of the mean value theorem (concerning the first derivative) for the zeroth derivative, i.e. the function itself, states that if g is a continuous function and $g(x_1) \neq g(x_2)$ for $x_1, x_2 \in \text{dom}(g)$, then for any $\lambda \in [0,1]$ there exists an x^* between x_1 and x_2[1] so that

$$f(x^*) = \lambda g(x_1) + (1-\lambda) g(x_2).$$

One more hint can be found in the footnote.[2]

Problem 3: An Unconstrained Optimization Problem

This problem is taken from Simon and Blume (1994), Ex. 17.1(i).

Consider the function $f : \mathbb{R}^2 \mapsto \mathbb{R}, (x_1, x_2)' \mapsto x_1^4 + x_1^2 - 6x_1x_2 + 3x_2^2$, and solve the problem of minimizing f over its domain. How can you argue that if there are local minimizers, at least one of them will be global?

Problem 4: Constrained or Unconstrained?!

This problem is taken from Simon and Blume (1994), Ex. 18.4(i).

Solve the budget-constrained utility maximization problem

$$\max x_1 x_2 \quad \text{s.t.} \quad x_1 + 4x_2 = 16.$$

Problem 5: Intuition dominates Computation

Solve

$$\max \frac{4}{3}x^2 + y + xz \quad \text{s.t.} \quad \|(x,y,z)\|_2 \leq 1$$

where $\|\cdot\|_2$ is the Euclidean norm of the $\mathbb{R}^3$.

If you go the formal way, i.e., checking first- and second order conditions, you get 6 solutions to the FOC. For either of these, you have to check two determinants of the Bordered Hessian, one for a 3×3 and one for a 4×4 matrix. Let's see if the intuition we had developed

[1] So $x^* \in [x_1, x_2]$ if $x_1 < x_2$ and $x^* \in [x_2, x_1]$ if $x_1 > x_2$.

[2] *Hint 3.* The name of the theorem giving us the result was already mentioned in hint 2.

for the constrained optimization approach can help us solve this problem in under 10 minutes (in class: without wiping the board).

Three simplifications and intuitions can be extremely helpful:

- If $\|(x, y, z)\|_2 < 1$, we can increase the objective varying marginally y.
- By norm non-negativity, $\|(x, y, z)\|_2 \leq 1$ is equivalent to $\|(x, y, z)\|_2^2 \leq 1^2$.
- By the value cost intuition, the associated Lagrangian multiplier of any strict local maximizer in the constrained problem should strictly exceed zero.

Problem 6: Two Constraints?

Solve

$$\max -4x + y + z \quad \text{s.t.} \quad 2x + y = 0,\ y^2 + z^2 = 1.$$