E600 MATHEMATICS

Problem Set 3: Multivariate Differentiation, Integration, and Taylor's Theorem

Fall Semester 2019, course taught by: Martin Reinhard

Problem 0: Important Objects of Differential Calculus

Consider $x_0 \in X \subseteq \mathbb{R}^n$, $f : X \mapsto \mathbb{R}$. What is the type of the following objects:

$$\frac{\partial f}{\partial x_1}(x_0), \qquad \frac{df}{dx}, \qquad \frac{\partial}{\partial x_1}, \qquad \nabla f, \qquad \frac{df(x_0)}{dx}$$

Also answer the following:

1. Does $\frac{df}{dx_1}$ exist? What about $\frac{\partial f}{\partial x}$?
2. What is the difference between $\frac{df}{dx}(x_0)$ and $\nabla f(x_0)$?

Problem 1: Function Invertability

Determine which of the following functions are invertible, and if not, which criterion (injectivity or surjectivity) fails. Can you restrict domain and/or codomain to ensure invertability?[1]

1. $f : \mathbb{N} \mapsto \mathbb{N}$, $n \mapsto n^4$
2. $f : \mathbb{R} \mapsto \mathbb{R}$, $x \mapsto x^2$
3. $f : \mathbb{R} \mapsto \mathbb{R}$, $x \mapsto \cos(x)$
4. $f : \mathbb{R} \mapsto \mathbb{R}$, $x \mapsto 3$

Problem 2: Convex and Quasi-Convex Functions

Use the Hessian Criterion to investigate whether the function

$$f(x,y) = \frac{1}{2}(y^3 + 2xy + x^2)$$

is (strictly) convex or concave on $\mathbb{R}^3$, and otherwise try to find the support restrictions on which one of the properties holds.

[1] We can always ensure surjectivity by restricting the codomain to $f[X]$. Here, let us focus only on more "crude" cuts, by which I mean restrictions to intervals.

Problem 3: Mean Value Theorem

Prove the Mean Value Theorem, i.e. show that for $f \in D^1(X)$, $X \subseteq \mathbb{R}$, for any $a, b \in X$ so that $a < b$, there exists $x_0 \in (a, b)$ so that

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

What does this imply for the existence of "critical values" of f on X, i.e. points $x \in X$ where $f'(x) = 0$?

Problem 4: Multivariate Chain Rule

a.) Deriving a more familiar Expression

Let $f : Y \times Z \mapsto \mathbb{R}$, $X = \times Z \subseteq \mathbb{R}^n$, i.e. consider a function f of the form $f(x) = f(y, z)$ where y and z are potentially vectors. Further, define $g(z) = f(y(z), z)$, so that we vary y in a specific fashion related to z. Using the multivariate chain rule, derive that

$$\frac{dg}{dz} = \frac{\partial f}{\partial y}\frac{dy}{dz} + \frac{\partial f}{\partial z}$$

or respectively, that for any $z \in Z$,

$$\frac{dg}{dz}(z) = \frac{\partial f}{\partial y}(y(z), z)\frac{dy}{dz}(z) + \frac{\partial f}{\partial z}(y(z), z)$$

b.) Application

Use either version of the multivariate chain rule to derive the marginal indirect utility of consumption for x_2 when $u(x_1, x_2) = \sqrt{x_1 x_2}$ and the budget constraint is $x_1 + 2x_2 = 9$.

Problem 5: Taylor Approximations and Cobb-Douglas

In this exercise, we are concerned with just how "good" Taylor approximations are and how "close" is close enough for a good approximation. In practice, the value of Taylor approximations is more that they yield tractable expressions, i.e. (matrix) polynomials, for differentiable but potentially arbitrarily complex functions. They are often used both in theoretical and quantitative macro and econometrics.

For what follows, consider a household with Cobb-Douglas utility over consumption c and leisure l, i.e. $u(c, l) = c^\alpha l^{1-\alpha}$. You can use that at points $(c, l) \neq (0, 0)$, this function is infinitely many times differentiable.

a.) Approximation of Order 1

Compute the Taylor approximation of order 1 to $u(c, l)$ at $(c_0, l_0) = (1, 1)$. For $\alpha = 1/2$, compare the approximated values for $(c, l) = (3/2, 1/2)$ and $(c, l) = (5, 4/5)$ to the true value of u.

b.) Approximation of Order 2

Compute the Taylor approximation of order 2 to $u(c,l)$ at $(c_0,l_0) = (1,1)$. For $\alpha = 1/2$, compare the approximated values for $(c,l) = (3/2,1/2)$ and $(c,l) = (5,4/5)$ to the true value of u.

Write down the first order Taylor expansion for a $\lambda \in (0,1)$.

Problem 6: Multivariate Integration

Consider an economy populated by a mass $[0,1]$ of firms that use capital k and labor l to produce output $y = f(k,l) = Ak^{\alpha}l^{1-\alpha}$, i.e. they use the same Cobb-Douglas production technology. Further, suppose that economy-wide output satisfies

$$Y = \int_{[0,1]\times[0,1]} f(k,l)d(k,l).$$

Amongst others, this relationship can be obtained from assuming that labor l and capital k are independently and uniformly distributed on $[0,1]$. However, it is not too important what this means here, it just ensures that the equality above holds.

Determine Y as a function of A and α. What do you conclude for the role of α, the relative importance of capital in the production process in terms of its relationship to Y?