E600 MATHEMATICS

Problem Set 2: Matrices, Equation Systems

Fall Semester 2019, course taught by: Martin Reinhard

Problem 1: The Space of Matrices is a Vector Space

Show that the space $\mathcal{M}_{n\times m}$ as defined in the lecture constitutes a vector space.

Problem 2: Matrix Operations

a.) Some Examples

Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -4 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -5 & 3 \\ 2 & 4 \end{pmatrix}.$$

Determine whether the following matrices exist, and if so, compute them:

1.	AB	2.	BA	3.	$B'A'$
4.	$BA+C$	5.	$AB+C$	6.	$(AB+C)'$

Hint: Be aware of the rules for transposition and matrix operations to take some shortcuts!

b.) Conformability of Matrices and Vectors

Let A be the matrix as in a.). What $n \in \mathbb{N}$ must we choose so that $x \in \mathbb{R}^n$ can be right-multiplied to A, i.e. as Ax? What about $A'x$?

Problem 3: Elementary Operations for 3×3 Matrices

Here, we convince ourselves again that the elementary operations really work in the way we introduced them: Consider the matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

Define the matrix E so as to

1. interchange rows 2 and 3 (call the matrix E_1),
2. multiply rows 1 and 3 with $\lambda = 5 \neq 0$ (call the matrix E_2),
3. subtract two times row 1 from row 2 (call the matrix E_3).

Multiply out EA for $E \in \{E_1, E_2, E_3\}$ and check that indeed, the respective operation is performed.

Problem 4: The Nullspace and the Dimension of the Solution Set

A key concept related to solving equation systems in matrix notation that we haven't touched in the lecture is the "Nullspace" of a matrix A, also called the kernel, defined as

$$\ker A = \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}.$$

It is straightforward to verify the subspace property since $A(\lambda x + \mu y) = \lambda Ax + \mu Ay$. Here, we deal with its relation to the set of solutions. It will allow us to more formally address our intuition of free variables through the fundamental theorem of Linear Algebra, a really powerful result that you should have seen at least once!

a.) Solutions and the Kernel

Show that if there exists a solution x^* to the equation system $Ax = b$ in matrix notation, then x^s is a solution if and only if there exists an $x_0 \in \ker A$ so that $x^s = x^* + x_0$.

Also answer the following:

1. What can you conclude for the dimension of the number of free dimensions in the problem?
2. Suppose that $B_K(A) = \{b_{K,1}, \ldots, b_{K,d}\}$, $d \in \{0, 1, \ldots, m\}$, is a basis of $\ker A$. How can you use $B_K(A)$ to represent the solutions of $Ax = b$?

b.) The Fundamental Theorem of Linear Algebra

The theorem tells us about the interrelation of the number of free dimensions and the rank: it states that for $A \in \mathbb{R}^{n \times m}$,

$$\dim(\ker A) = m - \text{rk } A.$$

Note that unique solutions to $Ax = b$ can only exist if $\dim(\ker A) = 0$ (recall a.)), which already rules out unique solutions if $n < m$, i.e. strictly more unknowns than equations.

Noting that $(1, 1, 1)'$ is a solution, apply the theorem to determine the number of free variables in $Ax = b$ when

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

and use a.) to characterize the set of solutions.

Problem 5: Matrix Inversion

Derive 2x2 rule for the inverse, i.e. show that

$$\text{If } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } \det(A) = ad - bc \neq 0, \text{ then } A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

To do so, either use the Gauss-Jordan algorithm or multiply out AA^{-1}. Can you invert C of Problem 2? If so, what is C^{-1}?

Problem 6: Linear Independence Tests

In my experience, the rank concept, and especially determination of the number of linearly independent vectors in a set, is an awkward concept to many students, perhaps because many courses only teach the theorem for testing linear independence, but not the uniformly applicable matrix method, so that one has to revert to case-specific solution approaches, which are oftentimes non-obvious. Long story short, I believe it is really important that you are (or more positively: you may gain alot from being) familiar with the matrix-based linear independence test. Thus, let us practice it!

Consider the following sets of vectors:

$$S_1 = \left\{ \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 13 \\ 37 \\ 16 \end{pmatrix} \right\}, \quad S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \\ 0 \end{pmatrix} \right\}, \quad S_3 = \left\{ \begin{pmatrix} -2 \\ 2 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 3 \\ 0 \end{pmatrix} \right\}.$$

Recall: to perform the test, bring the matrix of stacked column vectors to (generalized) triangular form and investigate the rank.

Problem 7: Eigenvalues, Definiteness and Invertability

Let $A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$. Determine the eigenvalues of A. What can you say about its definiteness? Is it invertible? How could you have checked invertability more directly?