E600 MATHEMATICS

Problem Set 1: Vector Spaces, Basis and Norms

Fall Semester 2019, course taught by: Martin Reinhard

General Remark: I do not emphasize set convexity and hyperplanes as much as previous years. You may have a look at Problem 4 from last year to get some practice for these concepts.

Problem 1: The Algebraic Structure of Vector Spaces

a.) Computing the Scalar Product

Compute $a \cdot b$ for

1. $a = (2,5,1)'$, $b = (1,1,3)' \in \mathbb{R}^3$
2. $a = (2,0,-3,4)$, $b = (9,-8,7,-6) \in \mathbb{R}^4$
3. $a = (2,0,-3,4) \in \mathbb{R}^4$, $b = (4,1,2) \in \mathbb{R}^3$

What do you tell your colleague who claims to have found $v \in \mathbb{R}^n$ so that $v \cdot v = -1$?

b.) Third Cancellation Law

Prove that in a real vector space $(X,+,\cdot)$, it holds for $\lambda, \mu \in \mathbb{R}$ and $x \in X$ that

$$(\lambda x = \mu x \wedge x \neq \mathbf{0}) \Rightarrow \lambda = \mu$$

Problem 2: Subspaces, Linear Dependence and Basis

a.) A Proper Subspace of $\mathbb{R}^3$

Prove that

$$S_2 := \{x = (x_1, x_2, x_3)' \in \mathbb{R}^3 : x_2 = 0\}$$

gives rise to a proper subspace of $\mathbb{R}^3$. What is its dimension?

Hint: use the linear combination definition of the subspace.

b.) A Proper Subspace of $\mathbb{F}$ and its Basis

From the lecture, we know that $\mathbb{F} = (F, +, \cdot)$ with $F = \{f : X \mapsto \mathbb{R}\}$, $X \subseteq \mathbb{R}$ is a real vector space, and we shown that $C^0(X)$, the set of continuous functions $f : X \mapsto \mathbb{R}$, is a subspace. It turns out that it is quite demanding to find a basis for $C^0(X)$. Here, we consider the set of second order polynomials for which a basis can be found easily and intuitively. Define

$$\mathbb{P}_2(X) := \{f : X \mapsto \mathbb{R} : (\exists a, b, c \in \mathbb{R} : f(x) = ax^2 + bx + c)\}$$

Show that this is a subspace of $\mathbb{F}$ and find a basis. What is its dimension?

Remark: In an analogous fashion, you can establish that polynomials of any order k constitute a subspace of dimension $k+1$.

c.) Bases

Think of three different bases for $\mathbb{R}^3$. Include $b_1 = (1,1,0)'$ in the second and $b_2 = (1,0,4)$ in the third.

Which of the following sets are bases of the $\mathbb{R}^2$ (S&B, Ex. 11.12)?

$$S_1 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix} \right\}, \quad S_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix} \right\}, \quad S_3 = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix} \right\}, \quad S_4 = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, e_2 \right\}$$

Problem 3: Norm and Metric in Vector Spaces

a.) The Binary Metric

Consider a real vector space $\mathbb{X} = (X, +, \cdot)$, and define the binary metric

$$d_B : X \times X \mapsto \mathbb{R}, d(x,y) = \mathbb{1}[x \neq y]$$

Show that the function indeed constitutes a metric.

b.) The Norms we use are actually Norms

Show that the 1-norm ("Manhattan"), the 2-norm ("Euclidean") and the infinity-norm ("Maximum") indeed define norms on the $\mathbb{R}^2$. Recall:

- 1-norm: $||x||_1 = |x_1| + |x_2|$
- 2-norm: $||x||_2 = \sqrt{x_1^2 + x_2^2}$
- infinity-norm: $||x||_\infty = \max\{|x_1|, |x_2|\}$

Sketch the unit-closed ball of these norms, i.e. $\bar{B}_1(0) = \{x \in \mathbb{R}^2 : ||x|| \leq 1\}$ *Hint:* For $|| \cdot ||_2$, you may use the **Cauchy-Schwarz inequality** for the Euclidean space $(\mathbb{R}^n, || \cdot ||_2)$, which states that for any $x, y \in \mathbb{R}^n$:

$$|x \cdot y| \leq ||x||_2 ||y||_2.$$

Remark: In an analogous fashion, you can establish that they constitute norms on any $\mathbb{R}^n$, $n \in \mathbb{N}$.

c.) Inverse Triangle Inequality

Let $(\mathbb{X}, \|\cdot\|)$ be a normed vector space. Show the inverse triangle inequality, that is, prove that

$$\forall x, y \in \mathbb{X} : \|x - y\| \geq \Big| \|x\| - \|y\| \Big|.$$

d.) A Relationship of p-Norms and the Maximum Norm

Consider the vector space $(\mathbb{R}^n, +, \cdot)$, $n \in \mathbb{N}$, and let $p < \infty$. Show that, for any $x \in \mathbb{R}^n$,

$$\|x\|_\infty \leq \|x\|_p \leq n^{1/p} \cdot \|x\|_\infty.$$

e.) Norm Continuity

Show that any norm is continuous. More formally: show that if $\mathbb{X} = (X, +, \cdot)$ is a vector space and $\|\cdot\|$ defines a norm on X, then it holds that for any $x_0 \in X$,

$$\forall \varepsilon > 0 \exists \delta > 0 : (x \in B_\delta(x_0) \Rightarrow \|x\| \in B_\varepsilon(\|x_0\|))$$

or equivalently

$$\forall \varepsilon > 0 \exists \delta > 0 : (\|x - y\| < \delta \Rightarrow \Big| \|x\| - \|y\| \Big| < \varepsilon).$$

Remark 1: This result is extremely helpful as a variety of useful arguments (e.g. pulling in the limit, preservation of sign in a neighborhood) apply to continuous functions.

Remark 2: If there is not enough time, we will discuss this in the lectures on Chapter 3.