E600 Mathematics

Chapter 1: Introduction to Vector Spaces

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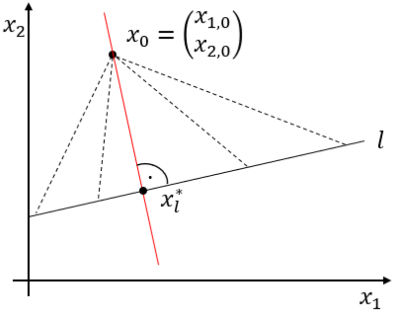
- We are ($\pm$) well-familiar with the mathematics of real numbers, and perhaps also with elements in $\mathbb{R}^2$ (two goods, consumption/leisure, $t = 0$ vs. $t = 1$, etc.)
- Here, we widely extend the range of objects with which we can do Math (addition, multiplication, etc.) in this familiar fashion, and also transfer graphical intuitions when a geometric picture is not available!
- Mostly interested in generalizing to $\mathbb{R}^n$ with many dimensions n, matrices and functions
- All generalizations have the same structure, the one of a **vector space**



- Which point x_l^* on the line l minimizes the distance to x_0?
- Easy to see: x_l^* must be such that the line connecting x_0 and x_l^* is orthogonal to l
- Pythagoras: $d^2 = \text{distance}^2 = (x_1 - x_{1,0})^2 + (x_2 - x_{2,0})^2$

$$\Rightarrow d = \sqrt{(x_1 - x_{1,0})^2 + (x_2 - x_{2,0})^2} = \|x - x_0\|_2$$

- Transfer to $\mathbb{R}^n$: points on a line that minimize the *Euclidean* distance to a point feature an orthogonal connection to it

- Vector of length n = ordered $n-$tuple of objects
- Row vs. column vector (convention: "vector" = column vector)
- Real vector of length 2: $x = (x_1, x_2)' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ where $x_1, x_2 \in \mathbb{R}$
- $\mathbb{R}^n := \{(x_1, \dots, x_n)' : (\forall i \in \{1, \dots, n\} : x_i \in \mathbb{R})\}, \; n \in \mathbb{N}$
- Function vector of length $n \in \mathbb{N}$: $f = (f_1, \dots, f_n)'$ where f_i, $i \in \{1, \dots, n\}$ are functions

$\Rightarrow$ Vectors can collect **any kind** of objects and be of **arbitrary** (including zero or infinite) length!

- Vector vs. set: $(2, 2, 3)' \neq (3, 2, 2)'$

- Consider the vectors $x = (0,4)', y = (2,-4)' \in \mathbb{R}^2$ (draw them!)
- Recall from high school: "Directionality and Magnitude"
    - $x = 4 \cdot (0,1)'$, $y = 6 \cdot (1/3, -2/3)'$
    - Fundamental directions of $\mathbb{R}^2$ (*axes*): $e_1 = (1,0)'$ and $e_2 = (0,1)'$
    - Any direction is a combination of them: e.g. $(1/3, -2/3)' = 1/3 \cdot e_1 + (-2/3) \cdot e_2$
- Ingredients:
    - Scalar multiplication: e.g. $4 \cdot (0,1) = (0,4) = x$ (scalar?)
    - Vector addition: e.g. $(1/3, 0)' + (0, 2/3)' = (1/3, 2/3)'$
    - Collection of fundamental directions: *(canonical) basis*
- **We don't need more for our generalization to arbitrary objects!!**

- How?
  - Start from any set X of real vectors, matrices, functions, whatever
  - Find "appropriate" definitions for scalar multiplication and vector addition, i.e. "similar" to their counterparts in the $\mathbb{R}^2$ (or the $\mathbb{R}^n$)
  - Find a set of directionalities that "span" the space $\mathbb{X} = (X, +, \cdot)$
- Why?
  - Transfer familiar mathematical and graphical approaches to complex objects
  - Far much easier representation: infinitely large set vs. two operators and a ("small") set of directions
  - Define further concepts in analogy to real vectors (e.g. distance of two functions)

2. The Algebraic Structure of Vector Spaces

Definition (Real Vector Space)

Let X be a set of vectors and $\mathbb{X} := (X, +, \cdot)$ be the collection of this set together with two operations, called vector addition and scalar multiplication, which associates to any scalar $\lambda \in \mathbb{R}$ and any $x \in X$ the vector $\lambda \cdot x$. Then, $\mathbb{X}$ is called a **vector space** *if the following properties hold:*

(i) *X is* **closed** *with respect to the operations: $\forall x, y \in X : x + y \in X$, and $\forall x \in X \forall \lambda \in \mathbb{R} : \lambda \cdot x \in X$.*

(ii) *Vector addition is* **commutative***: $\forall x, y \in X : x + y = y + x$*

(iii) *Vector addition is* **associative***: $\forall x, y, z \in X : x + (y + z) = (x + y) + z$.*

(iv) *There exists an "additive identity" element $\mathbf{0} \in X$ such that $\forall x \in X : x + \mathbf{0} = x$.*

(v) *Scalar multiplication is* **associative***: $\forall \lambda, \mu \in \mathbb{R}\ x \in X : \lambda \cdot (\mu \cdot x) = (\lambda\mu) \cdot x$*

(vi) *Scalar multiplication is* **distributive** *over vector and scalar addition:*

$$\forall \lambda \in \mathbb{R} \forall x, y \in X : \lambda(x + y) = \lambda x + \lambda y$$

$$\forall \lambda, \mu \in \mathbb{R} \forall x \in X : (\lambda + \mu)x = \lambda x + \mu x$$

(vii) $\forall x \in X : (1 \cdot x = x \wedge 0 \cdot x = \mathbf{0})$.

- Axiomatic definition due to the broad class of objects X may contain
- $+$ and $\cdot$ are called basis operations
- Closedness w.r.t. $+$ and $\cdot$ ensures that $x + y$ and $\lambda \cdot x$ are well-defined
- $+$ and $\cdot$ have two meanings, pay attention to the context ($\cdot$ gets a third one shortly)
- Definition has several further natural implications
  - Unique additive inverse: $\forall x \in X \exists ! x^- \in X : x + x^- = \mathbf{0}$
  - Cancellation laws for addition and scalar multiplication ($x+y = x+z$, $\lambda x = \lambda y$, $\lambda x = \mu x$ – but don't divide "by zero" (0 or $\mathbf{0}$)!)

Show that $\mathbb{F} := (F_X, +, \cdot)$ is a vector space when $X \subseteq \mathbb{R}$ and

- $F_X := \{f : X \mapsto \mathbb{R}\}$,
- $\forall f \in F_X, \lambda \in \mathbb{R} : (\forall x \in X : (\lambda \cdot f)(x) = \lambda \cdot f(x))$,
- $\forall f, g \in F_X : (\forall x \in X : (f + g)(x) = f(x) + g(x))$.

It's actually easier than it sounds...

- Cartesian Product
  - Recall $X \times Y$ from the introduction?
  - $\mathbb{X} = (X, +_X, \cdot_X)$ and $\mathbb{Y} = (Y, +_Y, \cdot_Y)$ vector spaces. Their Cartesian product is the vector space $\mathbb{X} \times \mathbb{Y} := (X \times Y, +, \cdot)$ where
    - $\forall (x_1, y_1), (x_2, y_2) \in X \times Y : (x_1, y_1) + (x_2, y_2) = (x_1 +_X x_2, y_1 +_Y y_2)$
    - $\forall (x, y) \in X \times Y \forall \lambda \in \mathbb{R} : \lambda \cdot (x, y) = (\lambda \cdot_X x, \lambda \cdot_Y y)$
  - Heavy notation, (relatively) simple concept. . .
- Scalar Product: function $\cdot : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}, (x, y) \mapsto x \cdot y = \sum_{i=1}^n x_i y_i$
  - Alternative notation: $\langle x, y \rangle$ or $x' y$
  - Third product: multiplication of vectors (pay attention!)
  - $x \cdot x = \sum_{i=1}^n x_i^2$ ("sum of squares")
  - Also called "dot product", "inner product" or "vector product"

- Let me take a break from my monologue and take 3 minutes to think about the following problems:
  1. If $x = (1, 2, 4)'$ and $y = (3, 0, 2)'$, what is $(2x) \cdot y$?
  2. Verbally or formally argue why the following are true for any $x, y \in \mathbb{R}^n$:
     1. $x \cdot y = y \cdot x$
     2. (Binomial Formula): $(x + y) \cdot (x + y) = x \cdot x + 2(x \cdot y) + y \cdot y$
     3. If $x \cdot x = 0$ then $x = 0$

     (Hint: think about the "sum of squares" property for the last two)

- Wrap up – thus far: location $\rightarrow$ basis operations

Outline

- Examples for vector spaces: all real sequences $\{x_n\}_{n\in\mathbb{N}}$ or all real-valued functions $f : X \mapsto \mathbb{R}$ with appropriate basis operations
- What if we only care about *convergent* sequences or *continuous* functions?
- Definition: $\mathbb{X} = (X, +, \cdot)$ real vector space and $Y \subseteq X$. If Y is closed under both basis operations of $\mathbb{X}$, then we call $\mathbb{Y} = (X, +, \cdot)$ a **(real) vector subspace** of $\mathbb{X}$
- Equivalent Definition: Y closed under linear combination
  - Linear combination (LC) of $x_1, \ldots, x_k \in X$ with coefficients $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$: $\sum_{j=1}^{k} \lambda_j \cdot x_j$
  - Closure of Y under LC: $\forall x, y \in Y \forall \lambda, \mu \in \mathbb{R} : \lambda x + \mu y \in Y$ (iteration)
  - Let's prove equivalence (two-step approach)
  - Arguably more convenient: just a single condition to remember

- Example 1: Function Space
  - Recall the space $\mathbb{F} = (F_X, +, \cdot)$ of all functions $f : X \mapsto \mathbb{R}$, $X \subseteq \mathbb{R}$
  - Consider $C^0(X) \subseteq F_X$ as the set of *continuous* functions $f : X \mapsto \mathbb{R}$
  - Let's prove that this is a subspace (recall: limit and continuity)!
- Example 2: Convergent real sequences as subspace of all real sequences (with appropriate operations)
- Counterexamples: the sets $\mathbb{N}^n$, $\mathbb{Z}^n$, $\mathbb{Q}^n \subseteq \mathbb{R}^n$ (e.g. $\pi \cdot \mathbf{1}$)
- Subspaces: Addition and intersection ✓, union X
  - $Y, Z \subseteq X$ give rise to subspaces (closed under LC)
  - $Y \cap Z$ and $Y + Z := \{x \in X : (\exists (y, z) \in Y \times Z : x = y + z)\}$ are subspaces, but generally not $Y \cup Z$ (why?)

- Recall: we wanted basis operations + fundamental directions to characterize vector space, still need the latter
- For our $\mathbb{R}^2$-example, indeed all $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ are LC's of the fundamental directions e_1 and e_2: $x = x_1 e_1 + x_2 e_2$, so that

$$\mathbb{R}^2 = \{x_1 e_1 + x_2 e_2 : x_1, x_2 \in \mathbb{R}\} = \{x : x \text{ is LC of } e_1, e_2\}$$

- Straightforward to extend to $\mathbb{R}^n$, right? Concept: *span*

Theorem (Span Operator and Generated Subspace)

Let $\mathbb{X} := (X, +, \cdot)$ be a real vector space, and let $Y \subseteq X$. We define

$$Span(Y) = \{z \in X : z \textit{ is LC of elements in } Y\}.$$

Then, $(Span(Y), +, \cdot)$ is a subspace of $\mathbb{X}$, called the subspace generated by Y or the **span** *of Y. It is the smallest subspace which contains Y.*

- Technical note of caution: $Span(Y)$ is a set, "the span" a space – people oftentimes use sets and spaces as synonyms, don't do this!
- With the span concept: $\mathbb{R}^2 = Span(\{e_1, e_2\})$
- Other objects: What is $Span(\{f, g\})$ when $f(x) = x + 1$, $g(x) = x^2 + 2$ for all $x \in \mathbb{R}$?
- Complication: some ambiguity as $Span(\{(2,0)', (0,2)'\}) = Span(\{(1,0)', (2,8)', (0,1/4)'\}) = Span(\mathbb{R}^2) = \mathbb{R}^2$
- Desire for efficiency: "basis" should be "smallest" set to *span* the $\mathbb{R}^2$
- To define smallest, we need a new (very important!) concept...

- Context: $\mathbb{X} = (X, +, \cdot)$ real vector space
- Linear dependence: $S \subseteq X$ set, $x \in X$ vector. x is linearly dependent of S if it is a LC of elements in S, or equivalently, $x \in Span(S)$
- Linear independence (LI): $x \notin Span(S)$; LI set $B \subseteq X$: no element linearly depends on the remaining set: $\forall b \in B : (b \notin Span(B \backslash \{b\}))$
  - E.g. $\{e_1, e_2\}$: $Span(\{e_1\}) = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\} \not\ni e_2$ and vice versa

Theorem (Testing Linear Independence)

A equivalent condition for the set of vectors $B = \{b_1, b_2, ..., b_k\}$ to be linearly independent is that

$$\sum_{j=1}^{k} \lambda_j b_j = \mathbf{0} \Rightarrow (\forall j \in \{1, ..., k\} : \lambda_j = 0). \tag{1}$$

The vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, ..., e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n$$

are linearly independent, because if $\lambda_1 e_1 + ... + \lambda_n e_n = 0$, i.e.

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + ... + \lambda_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The last vector equation implies that $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$.

General test: solve linear equation system (cf. Chapter 2);
$B = \{b_1, \ldots, b_n\} \subseteq \mathbb{R}^n$ is linearly independent if and only if the matrix $(b_1, \ldots, b_n)$ with the elements of B in columns has full rank

- Idea: can leave out linearly dependent vectors from "smallest set" as they can be represented by the remaining set
- Basis of vector space $\mathbb{X} = (X, +, \cdot)$: $B \subseteq X$ such that $X = Span(B)$ and B is LI set
  - Basis is **not unique**: $\{e_1, e_2\}$ vs. $\{(2,0)', (0,3)'\}$ vs. $\{(1,2)', (3,4)'\}$
  - $\mathbb{R}^n$: *canonical* basis $\{e_1, \ldots, e_n\}$ (fundamental directions) is unique
  - Generally: can require basis objects to have "unit length" (cf. norm)
- Dimension of vector space: cardinality (number of elements) in basis
  - Dimension is **unique**!
  - Intuition uniqueness: dimension determines number of independent directions; proof is quite involved (reference in script)

- We are done generalizing our one-slide intuition
  $\Rightarrow$ We can now efficiently and consistently represent arbitrary collections of objects (sequences, matrices, functions, etc)!
- ...so long as they constitute a space (="minimum requirement for basic operations")
  - We know how to check this for a general class of objects
  - ...as well as for sub-classes ("subspace")
- Now: exploit this uniform structure of vector spaces to generalize more concepts; we focus on
  - Distances
  - Continuity and Convergence
  - Convexity (and Concavity)

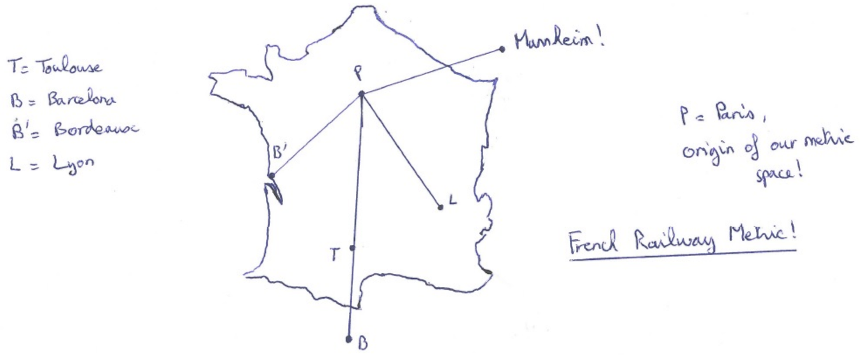
- First concept to generalize from the $\mathbb{R}^2$
- Consider Mannheim, which is, like Manhattan, organized in squares
- If you want to go watch a movie at Cineplex in P4 after we finish on Friday. . .
- Generally, what should we intuitively expect from a distance?
  1. Non-negative, and zero only if we don't have to move
  2. Symmetric: same distance from A to B and from B to A
  3. Detours increase the distance
- . . . that's exactly what Mathematicians think of a distance as well!

Definition (Metric)

$\mathbb{X} = (X, +, \cdot)$ *real vector space. Then, a* *function* $d : X \times X \mapsto \mathbb{R}$ *defines a* **metric** *on* X *if*

	Condition	*Name*
(i)	$\forall x, y \in X : d(x, y) \geq 0$, *and* $d(x, y) = 0 \Leftrightarrow x = y$	*non-negativity*
(ii)	$\forall x, y \in X : d(x, y) = d(y, x)$	*symmetry*
(iii)	$\forall x, y, z \in X : d(x, y) \leq d(x, z) + d(z, y)$	*triangle inequality*

- Ex.: Manhattan, Euclidean, binary (let's show Manhattan for $\mathbb{R}^2$)
- Metric space: $(\mathbb{X}, d)$ where $\mathbb{X} = (X, +, \cdot)$ and d metric on X
- Drawbacks: the measured distance may not be absolute but rather relative to...
  - the starting point: we may have $d(x, y) \neq d(x + z, y + z)$
  - scaling: we may have $d(\lambda x, \lambda y) \neq \lambda d(x, y)$ for $\lambda > 0$



$$d_{FR}(x,y) = \begin{cases} \|x-y\|_2 & \text{if } x = \lambda y \text{ for a } \lambda \in \mathbb{R}, \\ \|x\|_2 + \|y\|_2 & \text{else.} \end{cases}$$

- $\|\cdot\|$: "Euclidean norm" $\Leftrightarrow$ direct distance (formalized shortly)

Definition (Norm and Normed Vector Space)

$\mathbb{X} = (X, +, \cdot)$ *real vector space. Then, a function* $\|\cdot\| : X \mapsto \mathbb{R}$ *defines a norm on* X *if*

	Condition	*Name*
(i)	$\forall x \in X : \|x\| \geq 0$, *and* $\|x\| = 0 \Leftrightarrow x = \mathbf{0}$	*non-negativity*
(ii)	$\forall x, y \in X : \|x + y\| \leq \|x\| + \|y\|$	*triangle inequality*
(iii)	$\forall x \in X, \lambda \in \mathbb{R} : \|\lambda \cdot x\| = \lvert\lambda\rvert \cdot \|x\|$	*absolute homogeneity*

Then, we call $(\mathbb{X}, \|\cdot\|)$ *a* **normed vector space**.

- p-Norm on $\mathbb{R}^n$: $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$, Max. norm: $\|x\|_\infty = \max_{i \in \{1,\dots,n\}} |x_i|$
- Natural norm on $\mathbb{R}$: $\|x\| = |x|$ (equal to any $\|x\|_p$, including $p = \infty$!)
- Norm-induced metric: $d_N(x, y) = \|x - y\|$ (metric property see script)

- Why norm-induced metrics d_N?
  - d_N exhibits the following extra properties (why?):
    1. absolute homogeneity: $\forall x, y \in X \ \forall \lambda \in \mathbb{R} \ d(\lambda x, \lambda y) = |\lambda| d(x, y)$
    2. translation invariance: $\forall x, y, z \in X \ d(x + z, y + z) = d(x, y)$

    $\Rightarrow$ broader range of appealing properties!
  - Length/magnitude as distance from origin: $\|x\| = \|x - \mathbf{0}\| = d_N(x, \mathbf{0})$
- Economists typically consider **Euclidean Spaces** $(\mathbb{R}^n, d_N^2)$ with

$$d_N^2(x, y) = \|x - y\|_2 = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2} = \sqrt{(x - y) \cdot (x - y)}$$

  - Crucial importance in Econometrics: Least Squares Estimators
  - Geometrical intuition: direct distance
  - "The distance" usually means the Euclidean norm(-induced metric)

Let's continue our function space example...

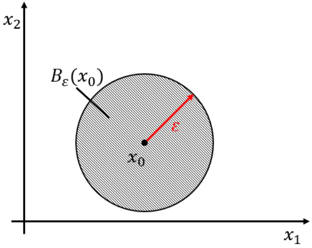
- How to define distance of $f(x) = 2\sin(x)$ and $g(x) = cos(x)$?
- Functions: *supremum* norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$
  - Supremum = "generalized maximum", introduced later
  - sup = max whenever max exists (counterex. (0,1); $\sup(0, 1) = 1$)
  - $\|f\|_\infty = 2$; $\|g\|_\infty = 1$
  - Distance (draw): $\|f - g\|_\infty = \max_{x \in X} |f(x) - g(x)| = 2$

Finally, a useful trick for norms ("inverse triangle inequality"):

$$\forall x, y \in X : \|x - y\| \geq \big|\|x\| - \|y\|\big|.$$

Proof: see Problem Set.

- Distance functions are fundamentally important for economics
  - Limits and Continuity of general functions are defined using them
  - Related set properties (open, closed, compact) are at the heart of optimization
  - Least squares estimators
  - ...
- Let's begin with the necessary definitions!



- Again: intuition from the $\mathbb{R}^2$
- Ball of radius $\varepsilon > 0$ around x_0: all points with distance to x_0 "smaller" than ε
  - strictly ($d(x, x_0) < \varepsilon$): open ball $B_\varepsilon(x_0)$
  - weakly ($d(x, x_0) \leq \varepsilon$): closed ball $\bar{B}_\varepsilon(x_0)$
  - "closed balls include the boundary, open balls do not"
- Two types of points: interior and boundary points ($int(A)$ vs. ∂A)
- Open set: only interior, no boundary points: $A = int(A)$
- Closed set: also includes all boundary points: $A = int(A) \cup \partial A$
- Bounded set: bounded distance of elements:
  $\exists x \in X \exists r < \infty : A \subseteq B_r(x)$
- Compact set: closed and bounded ("room with walls")

- Concepts are formally a bit tedious, see script for more detail
- Sets may be neither open nor closed (include boundary only partly, e.g. $[a, b)$) or both (no boundary, e.g. $\mathbb{R}$ or $\varnothing$)
- Open/closed interval in $\mathbb{R}$ is open/closed ball:

$$(a, b) = \left(\frac{a+b}{2} - \frac{b-a}{2}, \frac{a+b}{2} + \frac{b-a}{2}\right) = B_{\frac{b-a}{2}}\left(\frac{a+b}{2}\right)$$

- Compact = closed and bounded is actually a theorem (Heine-Borel)
- Compactness is fundamentally important for optimization

- ε-open Ball around x_0:

$$B_\varepsilon(x_0) \overset{\text{generally}}{=} \{x \in X : d(x, x_0) < \varepsilon\}$$
$$\overset{d \text{ norm-induced}}{=} \{x \in X : \|x - x_0\| < \varepsilon\}$$

- Interior point: $x \in int(A) \Leftrightarrow (\exists \varepsilon > 0 : B_\varepsilon(x) \subseteq A)$ (graphically?)
- Recall: A is open $\Leftrightarrow A = int(A)$; we usually only investigate openness (see next slide why)
- Frequently, proving the definition directly is unnecessarily tedious $\rightarrow$ how can we proceed more elegantly?

Theorem (Properties of Open and Closed Sets)

Consider a metric space $(\mathbb{X}, d)$. Then,

(o.i) $\varnothing$ and X are open in $\mathbb{X}$.

(o.ii) A set $A \subseteq X$ is open if and only if its complement $A^c = X \backslash A$ is closed.

(o.iii) The union of an arbitrary (possibly infinite) collection of open sets is open.

(o.iv) The intersection of a finite collection of open sets is open.

(c.i) $\varnothing$ and X are closed in $\mathbb{X}$.

(c.ii) A set $A \subseteq X$ is closed if and only if its complement $A^c = X \backslash A$ is open.

(c.iii) The union of a finite collection of closed sets is closed.

(c.iv) The intersection of an arbitrary (possibly infinite) collection of closed sets is closed.

Take-away: check complements and/or decompose into $\cup/\cap$ of simple sets!

Theorem (Closedness and Sequences)

Suppose that $\mathbb{X} = (X, +, \cdot)$ *is a real vector space, and let* $B \subseteq X$*. Then,* B *is closed if and only if, for any convergent sequence* $\{x_n\}_{n\in\mathbb{N}}$ *over* B*, i.e.* $\forall n \in \mathbb{N} : x_n \in B$*, it holds that* $\lim_{n\to\infty} x_n \in B$*.*

Theorem (Weak Inequalities and the Limit: Functions)

Suppose that $\mathbb{X} = (X, +, \cdot)$ *is a real vector space,* $f : X \mapsto \mathbb{R}$ *and* $g : X \mapsto \mathbb{R}$ *so that* $\forall x \in X$*:* $f(x) \leq g(x)$ *(in function notation:* $f \leq g$*). Let* $x_0 \in X$*, and suppose that* $\exists f_0, g_0 \in \mathbb{R}$ *so that* $\lim_{x\to x_0} f(x) = f_0$*,* $\lim_{x\to x_0} g(x) = g_0$*. Then, it holds that* $f_0 \leq g_0$*.*

Theorem (Weak Inequalities and the Limit: Sequences)

Suppose that $\mathbb{X} = (X, +, \cdot)$ *is a real vector space. Let* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *be convergent sequences over* X*, i.e.* $\forall n \in \mathbb{N} : x_n, y_n \in B$*, with limits* $x \in X$ *and* $y \in X$*, respectively. If* $\forall n \in \mathbb{N}$*, it holds that* $x_n \leq y_n$*, then, we also have* $x \leq y$*.*

Theorem (p-Norm and Maximum Norm)

For any $p \in \mathbb{N}\backslash\{0\}$, it holds that for any $x \in \mathbb{R}^n$, $\|x\|_\infty \leq \|x\|_p \leq n^{1/p} \cdot \|x\|_\infty$. (Proof: script)

Theorem (Checking Boundedness)

$(\mathbb{X}, d)$ metric space ($\mathbb{X} = (X, +, \cdot)$) where d is norm-induced, i.e. for $x, y \in X$, $d(x, y) = \|x - y\|$. Let $A \subseteq X$. Then, A is bounded if the norm is bounded on A, i.e. $\exists b < \infty : (\forall x \in A : \|x\| < b)$.

Theorem (Budget Set Compactness in the $\mathbb{R}^2$)

Consider the Euclidean space $\mathbb{R}^2$, and the set $B(y|p_1, p_2) := \{x = (x_1, x_2)' \in \mathbb{R}^2_+ : p_1x_1 + p_2x_2 \leq y\}$, the budget set with income $y \in \mathbb{R}$ given prices $p_1, p_2 \geq 0$. Then, the budget set is closed, and if $p_1, p_2 > 0$, the budget set is also bounded and thus compact.

- We want to generalize concepts from $\mathbb{R}^n$ to arbitrary vector spaces
- Can do so for convergence and continuity now!
- Convergence of a sequence:
  - Recall $\mathbb{R}$: real sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent with limit $x \in \mathbb{R}$ if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : (\forall n \in \mathbb{N}, n \geq N : |x_n - x| < \varepsilon)$$

  - Recall: $|\cdot|$ is the natural norm of the $\mathbb{R}$, so that equivalently

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : (\forall n \in \mathbb{N}, n \geq N : \|x_n - x\| < \varepsilon)$$

  - **Convergence in normed VS** $(\mathbb{X}, \|\cdot\|_X)$ with $\mathbb{X} = (X, +, \cdot)$: A sequence $\{x_n\}_{n\in\mathbb{N}}$ where $\forall n \in \mathbb{N} : x_n \in X$ ("sequence over X") is convergent with limit $x \in X$ if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : (\forall n \in \mathbb{N}, n \geq N : \|x_n - x\|_X < \varepsilon)$$

- Convergence of a function:
  - Recall: for a univariate, real-valued function, i.e. $f : X \mapsto Y$ with $X, Y \subseteq \mathbb{R}$, $f_a \in Y$ is the limit of f at $a \in X$ if

$$\forall \varepsilon > 0 \exists \delta > 0 : (\forall x \in X : (|x - a| \in (0, \delta) \Rightarrow |f(x) - f_a| < \varepsilon))$$

  - "Natural metric" justification: for a general function $f : X \mapsto Y$ where $X \subseteq (\mathbb{X}, \|\cdot\|_X)$ and $Y \subseteq (\mathbb{Y}, \|\cdot\|_Y)$ (bad notation!) $f_a \in Y$ is the limit of f at $a \in X$ if

$$\forall \varepsilon > 0 \exists \delta > 0 : (\forall x \in X : (\|x - a\|_X \in (0, \delta) \Rightarrow \|f(x) - f_a\|_Y < \varepsilon))$$

  - Equivalently using the open ball concept:

$$\forall \varepsilon > 0 \exists \delta > 0 : (\forall x \in B_\delta(x_0) \backslash \{x_0\} : \|f(x) - f_a\|_Y < \varepsilon)$$

- Continuity: just like before: $f(a) = \lim_{x \to a} f(x)$

- Script: more generally for any metric space (not "just" norm-induced distance) $\rightarrow$ test your understanding!
- Equivalent definition of continuity of f at x_0 (intuition?):

$$\forall \varepsilon > 0 \exists \delta > 0 : (\forall x \in B_\delta(x_0) : \|f(x) - f(x_0)\|_Y < \varepsilon)$$

- Sequence characterization and disproving like before: find $x_n \overset{n\to\infty}{\rightarrow} x_0$ with $f(x_n) \overset{n\to\infty}{\not\rightarrow} f(x_0)$ (non-existent or different limit)

4. Convexity of Sets

Motivation

- You may know convexity of functions; we deal with this later
- Here: convexity of *sets*
- Economists are not always fortunate enough to deal with spaces (e.g. budget set is not a space)
- how to preserve *most* of the structure?
- Recall space: *any* linear combination of elements contained
- Convex set: restrict attention to *convex* combinations

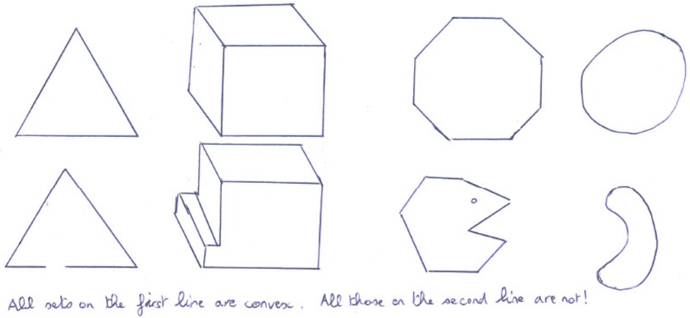
Definition (Convex Combination, Convex Set)

$\mathbb{X} = (X, +, \cdot)$ *real vector space. A convex combination* x^c *of the vectors* $x_1, \ldots, x_n \in X$ *is a linear combination* $x^c = \sum_{i=1}^n \lambda_i x_i$, *for which* $\forall i \in \{1, \ldots, n\} : \lambda_i \geq 0$ *and* $\sum_{i=1}^n \lambda_i = 1$.
A set $A \subseteq X$ *is convex if it contains all convex combinations of any two of its elements, i.e.* $\forall a_1, a_2 \in A \forall \lambda \in [0, 1] : \lambda a_1 + (1 - \lambda) a_2 \in A$.

- Iteration: A contains *any* convex combination
- 2 vectors x and y: $\{\lambda x + (1 - \lambda) y : \lambda \in [0, 1]\}$ is the line piece connecting x and y
- Multiple vectors: weighted average, area "between" them
- Graphical test for convexity in $\mathbb{R}^2$ and $\mathbb{R}^3$: line pieces fully contained in set

4. Convexity of Sets

Which Sets are Convex?



Convexity test – don't read the hand-written note!

- Convex sets in economics: e.g. budget sets (why?)

Recall *Span*(Y): generated subspace of Y. Now: "generated convex set" = *convex hull* (set of convex combinations of Y's elements):

$$Co(Y) = \left\{ x = \sum_{i=1}^{n} \lambda_i y_i : \left(y_1, \dots, y_n \in Y,\ \lambda_1, \dots, \lambda_n \geq 0 \text{ s.th. } \sum_{i=1}^{n} \lambda_i = 1 \right) \right\}$$

Proposition (Convexity-preserving Operations)

$\mathbb{X} = (X, +, \cdot)$ *real vector space. Then,*

(i) $\varnothing$ *and* X *are convex.*

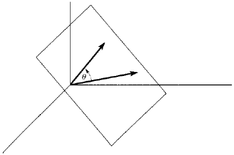
(ii) *if* $A \subseteq X$ *is convex, then so is* $\alpha A := \{\alpha \cdot a : a \in A\}$ *for any* $\alpha \in \mathbb{R}$.

(iii) *if* $A, B \subseteq X$ *are convex, then so is* $A + B := \{a + b : a \in A, b \in B\}$.

(iv) *if* $\{A_i\}_{i \in I}$ *is a (possibly infinite) collection of convex sets, then* $\bigcap_{i \in I} A_i$ *is convex.*

(Proof: see script)

Proposition may be helpful for proofs of convexity!



- In Euclidean space $(\mathbb{R}^n, \|\cdot\|)$ where $\|\cdot\|$ is the Euclidean norm, the radian angle θ of two non-zero vectors u, v satisfies

$$u \cdot v = \|u\| \cdot \|v\| \cos(\theta)$$

- Radian angle $\theta \in [0, 2\pi]$ (graphically?):
  - 90° (orthogonal): $\cos(\pi/2) = 0 \Rightarrow u \cdot v = 0$
  - Linearly dependent vectors
    - Same directionality ($x = \lambda y$, $\lambda \geq 0$): $\theta = 0 : 0^\circ$, $\theta = 2\pi : 360^\circ$ with $\cos(\theta) = 1 \Rightarrow u \cdot v = \|u\| \cdot \|v\| > 0$
    - Reversed directionality ($x = \lambda y$, $\lambda \leq 0$): $\theta = \pi : 180^\circ$ with $\cos(\theta) = -1 \Rightarrow u \cdot v = -\|u\| \cdot \|v\| < 0$

  $\Rightarrow$ generalizes orthogonality to $\mathbb{R}^n$!

$\Rightarrow$ Scalar products are geometrically important concepts!

- **Hyperplane** of $\mathbb{X} \subseteq \mathbb{R}^n$: set of vectors that share a certain scalar product with a fixed vector $a \in \mathbb{R}^n$, $a \neq \mathbf{0}$: For $b \in \mathbb{R}$,

$$H(a,b) = \{x \in X : a \cdot x = b\} = \{x \in X : \sum_{i=1}^{n} a_i x_i = b\}$$

  - $H(a,0)$ is the set of vectors orthogonal to a
  - The origin $\mathbf{0}$ has $\mathbf{0} \cdot a = 0$ for any $a \in \mathbb{R}^n$, so that $\mathbf{0} \in H(a,b) \Leftrightarrow b = 0$
  - A line in $\mathbb{R}^2$ $x_2 = mx_1 + b$ is a hyperplane: $H(\begin{pmatrix} -m \\ 1 \end{pmatrix}, b)$
  - A plane in $\mathbb{R}^3$ is also a hyperplane (see script)

  $\Rightarrow$ Powerful generalization of convex (why?) geometrical shapes to $\mathbb{R}^n$
- Hyperplanes in economics?

Theorem (Separating Hyperplane Theorem)

Let C and D be two convex and disjoint sets in a metric space $(\mathbb{X}, d)$ over the set X, i.e. $C \cap D = \varnothing$. Then, there exists $a \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $b \in \mathbb{R}$ such that $\forall x \in C : a \cdot x \leq b$ and $\forall x \in D : a \cdot x \geq b$. The hyperplane $\{x \in X : a \cdot x = b\}$ is called a separating hyperplane for the sets C and D.