E600 Mathematics

Welcome and Chapter 0: Fundamentals of Mathematics

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1. Logistics and Organization

What?

- Objectives
  - Review central mathematical concepts for economists
  - Get more familiar and comfortable with logic and notation
  - Know and be able to apply main theorems
  - Have a rough idea of why they are true (not main focus)
- Outline see script, per chapter:
  - $\approx$ 3 blocks (a 90 minutes) lecture
  - $\approx$ 1 block exercises

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When?

- Block course from Monday, August 26th to Friday, August 30th
- Usually: 09:30 to 17:00 with 1h lunch break (12:45 – 13:45) and two 15-min. coffee breaks
  - Wed: early lunch break 12:15 – 13:00, stop at 14:15 due to International Student Orientation Session
  - Wed: we likely meet again at 16:00 – 17:30
  - Fr: stop earlier (15:45) due to International Student Welcome Session

1. Logistics and Organization

Why?

- Why are you here?
  - No "self-motivated" course, objective is to help your Master's studies (especially first semester)!
  - Mannheim has a strong quantitative orientation $\rightarrow$ Mathematics is especially important here
  - Logical reasoning and mathematical methods are crucial for any economist
- Why am I here (and who am I)?
  - 3rd year PhD student at CDSE (talk to me if you are interested in the Research courses!), Math minor in undergrad in Mannheim
  - Don't hesitate to ask questions
    - during or after class
    - via mail: mareinhard@mail.uni-mannheim.de
    - no scheduled office hours, individual meetings can be arranged via mail

1. Logistics and Organization

How? 1/2

- Not mandatory, no exam, but: the more you learn here, the easier the first (and following) semesters will be!
- Course material: Script, slides and problem sets
  - Script meant as Math companion for your Master's (consistent notation and proofs that use only known concepts)
  - Script is self-sufficient resource, I mostly tell you what is written there
  - Problem sets to practice
  - All course material is uploaded on Ilias, script is also available at `https://www.vwl.uni-mannheim.de/media/Fakultaeten/vwl/Dokumente/E600_Script.pdf`
- Typical structure: problems in first block, lecture in blocks 2-4

  $\Rightarrow$ try to review and have a look at the problem sets in the evening!

1. Logistics and Organization

How? 2/2

- Mathematical level of content
  - "Intermediate" orientation: Proof > **Concept** > Application
  - Applications will be discussed especially in problem sets
  - Some proofs to increase your familiarity, but not main focus
- Main objective of course: multivariate generalization of familiar concepts, (graphical) intuition
- Key resources: books by Simon and Blume, and de la Fuente (see script)

2. Fundamentals: Why Math?

aka "Why not a review of Micro/Macro/Econometrics?"

- Let's briefly recap Chapter 0 (that you have already read, right?)
- Economists (among many other professions) put great emphasis on logical, rational reasoning – math is the purest form of this
- Economic investigations: mathematical model or statistical application $\rightarrow$ heavy reliance on Math (especially: optimization)
- Much heterogeneity in mathematical rigor of undergrad econ programs
- Math is actually less messy than the real world
  - No conceptual ambiguity: "fair economic system" vs. "Derivative of a given function f"?
  - Do the things even exist?
  - Statements are either 100% true or 100% false and can be proved/disproved with certainty

Vocab and Grammar of Mathematics 1/3

- Math is “like a language” very similar to normal English
- but: more symbols, some slightly different meanings
- Vocabulary: words + notation
- Grammar: rules that determine whether statements are meaningful
- Most important “sentences”: quantifiers
  - Existential quantifier $\exists \ldots$
  - Universal quantifier $\forall \ldots$

3. Fundamentals: Language and Logic

Vocab and Grammar of Mathematics 2/3

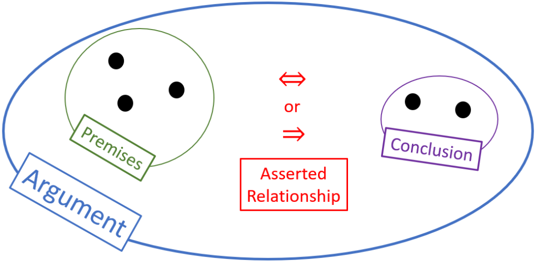
Basics and Quantifiers		*Logical Statements*	
Symbol	Meaning	Symbol	Meaning
$\exists$	there exists	$\Rightarrow$	implies
$\exists!$	there exists exactly one	$\Leftrightarrow$	is equivalent to
$\nexists$	there does not exist (any)	$\Leftarrow$	is implied by
$\forall$	for all	$\wedge$	logical "and"
$:$	which/for which/such that	$\vee$	logical "or"
	(alternatively: "it holds that")	$\neg$	logical "not"
$\in$	element of	$(\dots)$	delimiters of statement
$\notin$	not an element of		

Example (Def. continuity at $x_0 \in \mathbb{R}$):

$$\forall \varepsilon > 0 : (\exists \delta > 0 : (\forall x \in \mathbb{R} : (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon)))$$

- Miscellaneous comments
  - Being 100% formally correct requires the use of a lot of brackets and colons, they are left away if the statement is still clear
  - Typically, telling logical "or" and "and" apart requires most practice
  - The logical "or" ($\vee$) includes the English "and", so that e.g. $(x > 4 \vee x > 2)$ is true both for $x = 3$ and $x = 5$
  - $\geq$: "greater than" vs. $>$: "strictly greater than", e.g. "x is greater than x" would typically be true
  - Negations usually have shorthand notation
- Statements: meaningful (grammar) and true (content)?

Bananas blue	$\forall x \in \mathbb{R} : x > y$
Bananas are blue	$\forall x \in \mathbb{R} : (\forall y \in \mathbb{R} : x > y)$
Bananas are yellow	$\forall x \in \mathbb{R} : (\exists y \in \mathbb{R} : x > y)$



- Black dots: individual statements
- Argument = statement about relationship of statements
- Classic argument: 2 premises imply one conclusion
- Ex.: (P1: All Germans like beer, P2: I am German ⇒ C: I like beer)

- Arguments can be more complex:
  - Equivalence: (P1: I was born blind, P2: My blindness has never been healed, C: I have always been blind)
  - More than two premises, multiple conclusions (= conclusion including $\vee$ or $\wedge$, e.g. $(x > 5 \wedge f(x) > 10)$)
- Meaningful and true = "valid and sound"
  - valid: asserted relationship is correct
  - sound: the premises are true
  - Ex. 1: (Berlin is the Chinese capital $\wedge$ China is part of Europe) $\Rightarrow$ Berlin is a European capital
  - Ex. 2: (Berlin is the German capital $\wedge$ Germany is part of Europe) $\Rightarrow$ Berlin is a European capital
- Mathematical proof: establish non-obvious validity of argument

- Deduction: conclusion from general truth to particular case
  - P1: All bears like honey, P2: Pooh is a bear, C: Pooh likes honey
  - general: $\forall b \in$ "bears": $b \in$ "likes honey", and therefore...
  - particular: Pooh $\in$ "bears" implies Pooh $\in$ "likes honey"
- Induction: conclusion from all particularities to general case
  - Ask every German if he likes beer, if all say yes, the conclusion is that all Germans like beer
- Inductive proof: consider "every $n \in \mathbb{N}$ separately" to prove statement for all natural numbers
  - Inductive base: prove statement for smallest number ($n = 0$ or $n = 1$)
  - Inductive assertion: statement true for fixed $n - 1$
  - Inductive step: assertion implies that statement is true for n
  - Why does this work?

4. Fundamentals: Arguments and Reasoning

Necessary and Sufficient Conditions

- Script example: recession = (strictly) negative GDP growth for two consecutive quarters
  - Necessary conditions for recession (examples):
    - Average growth of last two quarters was negative
    - Growth was negative in last quarter
  - Sufficient conditions for recession (examples):
    - Growth at -2% during last two quarters
    - Negative growth lasting one year
  - Equivalent condition: necessary and sufficient
- Very important for optimization
- Final note of caution: Arguments do not address causation (e.g. Pooh, Germans)!

5. Fundamentals: Sets

Definition of Sets

- Set = collection of distinct objects ("elements")
  - $\{1, 2, \pi\}$ vs. $\{1, 1, \pi\}$ vs. $\{n \in \mathbb{N} : n > 10\}$
  - Elements can be sets, matrices, functions, whatever
  - Subset: $A \subseteq S \Leftrightarrow (x \in A \Rightarrow x \in S)$
  - Proper Subset: $A \subset S \Leftrightarrow (A \subseteq S \wedge (\exists x \in S : x \notin A))$
- Intervals: special subsets of $\mathbb{R}$
  - $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
  - (right-/left-) closed and open interval: $(a, b]$, $[a, b)$, $[a, b]$
- "All-set" and "nothing-set"
  - "All-encompassing" universal superset X: $\forall A : A \subseteq X$
  - "Encompassed-by-all" empty set $\varnothing$: $\forall A : \varnothing \subseteq A$

- Relations between sets A and B
  - Equal vs. disjoint
  - Complement A^c of A: $A^c = \{x \in X : x \notin A\}$
- Operations
  - Union $A \cup B$, Intersection $A \cap B$, Set difference $A \backslash B$ (circles)
  - Power set $\mathcal{P}(A) = \{S \subseteq X : S \subseteq A\}$
  - Index sets for *sets/sequences of sets*: $\bigcup_{i \in I} A_i$, $\bigcap_{i \in I} A_i$ (e.g. $I = \mathbb{N}$; what does pairwise disjoint mean?)

- Usual notation: $f : X \mapsto Y, x \mapsto y = f(x)$
  - X: domain, Y: codomain
  - Relates each $x \in X$ to exactly one (not necessarily unique) $y \in Y$

    $\Rightarrow f$ defines a relation on $X \times Y = \{(x, y) : x \in X, y \in Y\}$
  - $G(f) = \{(x, y) \in X \times Y : y = f(x)\} = \{(x, f(x)) : x \in X\}$ ("graph")
  - The graph $G(f)$ of f defines f as a relation!
  - Relations are sets!
- Concept: f is a function, $f(x)$ is not! (rather: element of codomain)

- Consider $f : X \mapsto Y$ (e.g. $f : \mathbb{R}_+ \mapsto \mathbb{R}, x \mapsto \sqrt{x}$)
  - Image of $A \subseteq X$, preimage of $B \subseteq Y$ ($A = [1, 4]$, $B = [4, 16)$)
  - $g : Y \mapsto Z$; composition $g \circ f$
  - Inverse function f^{-1} of f: $f^{-1} \circ f = f \circ f^{-1} = Id$
  - The preimage and the inverse function are fundamentally distinct objects!

7. Fundamentals: Limits and Continuity in $\mathbb{R}$

- Sequence: $\lim_{n\to\infty} x_n = x \Leftrightarrow |x_n - x| \overset{n\to\infty}{\to} 0$
- Function $f : X \mapsto \mathbb{R}$ where $X \subseteq \mathbb{R}$:

$$\lim_{x\to a} f(x) = f_a \in \mathbb{R} \quad \Leftrightarrow \quad |f(x) - f_a| \underset{x\neq a}{\overset{x\to a}{\to}} 0$$

- Notation: divergence $\lim_{x\to a} f(x) = \pm\infty$ (e.g. $f : (0,\infty) \mapsto \mathbb{R}, x \mapsto 1/x$)
- Continuity of f at $a \in X$: $\lim_{x\to a} f(x) = f(a)$
  - Examples for discontinuity: $f(x) = \mathbb{1}[x > 0]$, $f(x) = \mathbb{1}[x = 0]$
  - Characterization: left and right limit exist at a and are equal to $f(a)$
  - Prove continuity: formal definition, left/right limits
  - Disprove continuity: left/right limits, sequence $a_n \to a$ where $f(a_n) \not\to f(a)$

- lim . . . can be pulled into any continuous expression (including sums, products)
- Differentiable Functions: L'Hôpital's rule
  - e.g. $\lim_{x\to 0} \frac{\sin(x)}{x}$
- Sequences: Sandwich Theorem. Suppose $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \bar{x}$ and

$$\forall n \geq N \in \mathbb{N} : y_n \leq x_n \leq z_n.$$

  Then, $\lim_{n\to\infty} x_n = \bar{x}$.
  - e.g. $\lim_{n\to\infty} -\frac{1}{n^2+4n+25}$